International Journal of Theoretical Physics, Vol. 38, No. 1, 1999

Minimal Irreversible Quantum Mechanics: An Axiomatic Formalism

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Received August 7, 1998

An axiomatic formalism for a minimal irreversible quantum mechanics is introduced. It is shown that a quantum equilibrium and the decoherence phenomenon are consequences of the axioms and that Lyapunov variables, exponential survival probabilities, and a classical conditional never-decreasing entropy can be defined.

1. INTRODUCTION

Let us consider the function $y = f(x) = x^2$. Can we say if this function is an even function or an odd function? The primary (but incorrect) answer would be that it is an even function. This answer is wrong because, in order to define a function properly, we must also define its domain of definition D and its range R, namely

$$f(x) = y$$
$$f: \quad D \to R \tag{1.1}$$

Then if $y = f(x) = x^2$ is defined as f: $\mathbb{R} \to \mathbb{R}_+$, it is an even function. But if it is defined as f: $\mathbb{R}_+ \to \mathbb{R}_+$, the function is neither even nor odd. The moral of this story is that when we speak about the symmetry of a function necessarily we must define its domain of definition and its range; if not, what we may say could be meaningless.

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Let us now consider the Schrödinger equation

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle \tag{1.2}$$

and its solution

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \tag{1.3}$$

From what we have just learnt, the question, "Is the set of the time evolutions obtained from the Schrödinger equation time-reversible or invertible?" [1, 2], *has no meaning* if we do not define the domain of definition and the range of the states $|\psi\rangle$ in Eq. (1.3), i.e., the space where these vectors live. If we choose a Hilbert space $\mathcal{H}$ the set of time evolutions of Eq. (1.3) is time-symmetric and each evolution is invertible. But if we make a different choice the set can become time-asymmetric and each time evolution can become noninvertible. Let us explain why this is so.

Let K be the Wigner time-inversion operator. Hilbert space is invariant under time inversion, namely

$$K\colon \quad \mathcal{H} \to \mathcal{H} \tag{1.4}$$

But we can choose a non-time-reversal-invariant space as the space of physically admissible states; let it be ϕ_-, such that

$$K\colon \quad \phi_- \to \phi_+ \neq \phi_- \tag{1.5}$$

and then within this space the set of time evolutions will turn out to be time-asymmetric and each evolution noninvertible, as is shown in the literature (precisely in almost all the references of this paper) and as we will also demonstrate below.

In this minimal way we can obtain a natural irreversible quantum mechanics. The aim of this paper is to sketch, using the results of the authors quoted in the references and our own results, an axiomatic formalism for this theory which may have two possible advantages over ordinary quantum mechanics:

(i) The universe is clearly time-asymmetric. The new theory may describe the real universe better than the usual one. We shall further discuss this possibility in Section 16.

(ii) The new theory has more powerful spectral decompositions, which makes the study of decaying processes easier.

Let us rephrase what we have said, using physical language: If we forget the time-asymmetric weak interaction (as is usual in this kind of research, since the weak interaction is so weak that it is difficult to see how it can explain the macroscopic time asymmetry [3]), the time-asymmetry problem can be stated in the following question:

How can we explain the obvious time asymmetry of the universe and most of its subsystems if the fundamental laws of physic are time-symmetric?

There are only two causes for asymmetry in nature: either the laws of nature are asymmetric or the solutions of the equations of the theory are asymmetric. As time asymmetry is not an exception, the answer is contained in the question itself: If the laws of nature are time-symmetric, essentially the only way we have to explain the time asymmetry of the universe is to postulate that the state of the universe, or, more generally, the space of physically possible solutions of the universe evolution equations, is not time-reversal invariant, namely to use the second cause of asymmetry [2, 4]. In this paper we explore this possibility, using an axiomatic framework.

Moreover, certainly the best way to explain a physical idea is to construct an axiomatic structure because, having this structure, somehow we can see the whole idea, even if we cannot foresee all its consequences. Analogously, it is easier to criticize an idea when it is presented in an axiomatic language. So we believe that this paper can clarify some issues of the problem of time asymmetry.

The paper is organized as follows: In Section 2 we define the space and the notation we will use. In Section 3 the analytic continuation of the solutions is studied. In Section 4 density matrices and Liouville space are introduced. In Section 5 the space for the observables is chosen. In Section 6 the axioms of the theory are stated. In Sections 7 and 8 the main consequences of the axioms are obtained. In Section 9 time asymmetry and irreversibility are studied. In Section 10 we show how the Schrödinger and Heisenberg pictures work in the new formalism. In Section 11 quantum equilibrium and decoherence are obtained. In Section 12 it is shown that the norm and the energy are conserved and how Lyapunov variables appear. In Section 13 entropy is defined. In Section 14 the thermalization phenomenon is studied. In Section 15 the global nature of time asymmetry is considered. In Section 16 the Reichenbach diagram is presented. In Section 17 other results are listed. In Section 18 we draw our main conclusions. An appendix completes the paper.

2. DEFINITION SPACE

Let us consider a quantum system with a free Hamiltonian H_0, endowed with a discrete plus a continuous spectrum, namely such that

$$H_0 \big| E_n^{(0)} \rangle = E_n^{(0)} \big| E_n^{(0)} \rangle$$
$$H_0 \big| E^{(0)} \rangle = E^{(0)} \big| E^{(0)} \rangle \qquad (2.1)$$

where $n = (0, 1, 2, \ldots, N_0)$, $0 \leq E^{(0)} \leq \infty$, $E_n^{(0)} \geq 0$. The total Hamiltonian will be $H = H_0 + V$, and the perturbation will be such that some bound states of the discrete spectrum become complex poles, namely we will have

$$H|E_n\rangle = E_n|E_n\rangle$$
$$H|E\rangle = E|E\rangle \tag{2.2}$$

where $n = (0, 1, 2, \ldots, N)$, $0 \leq E \leq \infty$, and $N < N_0$ (in almost all cases, for simplicity and in order to fix ideas, we will consider that $N = 0$, and therefore there is only one discrete *ground state*; a more general case will be considered in Section 14). Here $\{|E_n\rangle, E\pm\rangle\}$ is a basis of the corresponding Hilbert space $\mathcal{H}$ (e.g., $|E\pm\rangle$ can be the Lippmann–Schwinger retarded or advanced bases $\{|\omega_\pm\rangle\}$ of ref. 5, and

$$\langle E_n|E_m\rangle = \delta_{nm}, \qquad \langle E\pm|E'\pm\rangle = \delta(E - E'), \qquad \langle E\pm|E_n\rangle = 0 \tag{2.3}$$

$$I = \sum_{n=0}^{n=N} |E_n\rangle\langle E_n| + \int_0^\infty |E\pm\rangle\langle E\pm|\, dE \tag{2.4}$$

$$H = \sum_{n=0}^{n=N} E_n|E_n\rangle\langle E_n| + \int_0^\infty E|E\pm\rangle\langle E\pm|\, dE \tag{2.5}$$

Let Ξ be the vector space of all possible linear combinations of the basis $\{|E_n\rangle, |E\pm\rangle\}$ vectors [6], so if $|\psi\rangle \in \Xi$, then

$$|\psi\rangle = \sum_{n=1}^{n=N} \psi_n|E_n\rangle + \int_0^\infty \psi_\pm(E)|E\pm\rangle\, dE \tag{2.6}$$

where neither ψ_n nor $\psi(E)$ has any peculiar property.

Let K be the Wigner time inversion operator [1]; therefore

$$K|E_n\rangle = |E_n\rangle, \qquad K|E\pm\rangle = |E\mp\rangle \tag{2.7}$$

(for the continuous spectrum the Lippmann–Schwinger advanced and retarded bases have this property). Then, as K is antilinear,

$$K|\psi\rangle = \sum_{n=1}^{n=N} \psi_n^*|E_n\rangle + \int_0^\infty \psi_\pm^*(E)|E\mp\rangle\, dE \tag{2.8}$$

To find an irreversible quantum mechanics we must define a subspace ϕ_- of Ξ such that

$$K\colon \quad \phi_- \to \phi_+ \neq \phi_- \tag{2.9}$$

namely a subspace which is not invariant under time inversions.

In our opinion there is a unique way to define space ϕ_- [7–10]. In fact it is completely reasonable to ask that ϕ_- have some logical properties, namely that $\phi_- \subset \mathcal{H}$ (i.e., $\psi_n \in l^2$, $\psi_\pm(E) \in L^2[0, \infty)$), ϕ_- must be dense in $\mathcal{H}_-$ (the outgoing state subspace of $\mathcal{H}$), and its topology must be a nuclear one. Precisely, we will define the spaces $\mathcal{H}_-$ and ϕ_- as

$$\begin{aligned} |\psi\rangle \in \mathcal{H}_- &\Leftrightarrow \psi_+(E) \in H^2_-\big|_{\mathbb{R}_+} \\ |\psi\rangle \in \phi_- &\Leftrightarrow \psi_+(E) \in \mathcal{S} \cap H^2_-\big|_{\mathbb{R}_+} = \mathcal{S}_- \end{aligned} \tag{2.10}$$

where $\mathcal{S}$ is the Schwarz class function space (this choice allows us to perform the derivative to any order) and H^2_- is the space of Hardy class function from below [5] (this choice introduces causality into our theory [10]). Nevertheless other choices have been used [11, 12].

As $\phi_- \subset \mathcal{H}_-$ we have the Gel'fand triplet:

$$\phi_- \subset \mathcal{H}_- \subset \phi_-^\times \tag{2.11}$$

where $\phi_-^\times$ is the space of antilinear functionals F over ϕ_-, such that

$$F[\psi] = \langle\psi|F\rangle = \langle F|\psi\rangle^* \tag{2.12}$$

This will be the main arena of all our calculations. but, as we will see, we must also use the time-inverted objects. Precisely, the spaces $\mathcal{H}_+$ and ϕ_+ defined as

$$\begin{aligned} |\psi\rangle \in \mathcal{H}_+ &\Leftrightarrow \psi_-(E) \in H^2_+\big|_{\mathbb{R}_+} \\ |\psi\rangle \in \phi_+ &\Leftrightarrow \psi_-(E) \in \mathcal{S} \cap H^2_+\big|_{\mathbb{R}_+} = \mathcal{S}_+ \end{aligned} \tag{2.13}$$

where H^2_+ is the Hardy class from above, and the Gel'fand triplet is

$$\phi_+ \subset \mathcal{H}_+ \subset \phi_+^\times \tag{2.14}$$

It is easy to see that the spaces ϕ_- and ϕ_+ satisfy (2.9).

We close the section with three observations.

(i) The Hamiltonian H must be time independent, since our aim is to define an arrow of time in a closed system. In fact, a realistic arrow of time must be defined using the whole universe as the system [13] (open systems will be considered in Section 14).

(ii) At first sight one might think that with the method we are about to propose one can define time asymmetry in a noninteracting system like a free particle. This is not so since, even if the resulting free particle theory would formally be time-asymmetric, the entropy will not grow. In fact, the entropy will only grow, as we will see, if we have a nontrivial S-matrix with complex poles, which is not the case of a trivial free particle.

(iii) ϕ_- is dense in $\mathcal{H}_-$, so, if someone would say that the "real" physical states are those of $\mathcal{H}_-$, we can answer that any one of these states can be approximated, as close as we wish, with a state of ϕ_-. So, on physical measurement grounds, the states of both spaces are indistinguishable. Nevertheless the two spaces have different kinds of topologies.

3. ANALYTIC CONTINUATIONS

Let us consider a scattering experiment using the Hamiltonian H and let $\{|\omega_+\rangle\}$ be the Lippmann–Schwinger basis (all objects related to this basis will be labeled with ω instead of the E used in the equations of the last section), then we know that

$$\sum_{n=0}^{N} |\omega_n\rangle\langle\omega_n| + \int_0^\infty |\omega_+\rangle\langle\omega_+| \, d\omega = I \tag{3.1}$$

where the $|\omega_n\rangle$ are the eventual stable bound states. Then

$$\langle\varphi|\psi\rangle = \sum_{n=0}^{N} \langle\varphi|\omega_n\rangle\langle\omega_n|\psi\rangle + \int_0^\infty \langle\varphi|\omega_+\rangle\langle\omega_+|\psi\rangle \, d\omega \tag{3.2}$$

Let z_n be the real and complex poles of the corresponding S-matrix. Then, using a simple analytic continuation of Eq. (3.2), it can be demonstrated [5] that if $|\psi\rangle \in \phi_-$ and $|\varphi\rangle \in \phi_+$, the inner product $\langle\varphi|\psi\rangle$ (which is well defined since both vectors belong to $\mathcal{H}$) reads

$$\langle\varphi|\psi\rangle = \sum_{n=0}^{N_0} \langle\varphi|\overline{f_n}\rangle\langle\tilde{f}_n|\psi\rangle + \int_\Gamma \langle\varphi|\overline{f_z}\rangle\langle\tilde{f}_z|\psi\rangle \, dz \tag{3.3}$$

where Γ is a curve that begins at O and goes to $+\infty$ of the real axis under all the poles of the lower half-plane. Also, making the Nakanishi trick [2, 19], we can obtain

$$\langle\varphi|\psi\rangle = \sum_{n=0}^{N_0} \langle\varphi|\overline{f_n}\rangle\langle\tilde{f}_n|\psi\rangle + \int_0^\infty \langle\varphi|\overline{f_\omega}\rangle\langle\tilde{f}_\omega|\psi\rangle \, d\omega \tag{3.4}$$

where there is a term in the sum for each pole of the S-matrix, precisely, for each complex pole and each real pole corresponding to the bound states of the sum of Eq. (3.1). Analogously, it can be demonstrated that

$$\langle\varphi|H|\psi\rangle = \sum_{n=0}^{N_0} z_n\langle\varphi|\overline{f_n}\rangle\langle\tilde{f}_n|\psi\rangle + \int_0^\infty \omega\langle\varphi|\overline{f_\omega}\rangle\langle\tilde{f}_\omega|\psi\rangle \, d\omega \tag{3.5}$$

(see also ref. 14), where $|\overline{f_n}\rangle, |\overline{f_\omega}\rangle \in \phi_+^\times$, $|\tilde{f}_\omega\rangle, |\tilde{f}_\omega\rangle \in \phi_-^\times$, and in particular $|\tilde{f}_n\rangle = |\omega_n\rangle$ (of $0 \le n \le N$) and $|\tilde{f}_\omega\rangle = |\omega_+\rangle$ [Eq. (44), ref. 15].

Also, if z_n is real, it is the eigenenergy of a bound state; and if z_n is complex, it is a pole of the S-matrix.

Therefore any $|\psi\rangle \in \phi_-$ reads

$$|\psi\rangle = \sum_{n=0}^{N_0} |\overline{f_n}\rangle\langle \tilde{f}_n|\psi\rangle + \int_0^\infty |\overline{f_\omega}\langle \tilde{f}_\omega|\psi\rangle \, d\omega \tag{3.6}$$

in a weak sense (namely premultiplied by any $\langle\varphi| \in \phi_+$) and $\langle \tilde{f}_\omega|\psi\rangle \in \mathcal{S}_-$.

Analogously,

$$H|\psi\rangle = \sum_{n=0}^{N_0} z|\overline{f_n}\rangle\langle \tilde{f}_n|\psi\rangle + \int_0^\infty \omega|\overline{f_\omega}\langle \tilde{f}_\omega|\psi\rangle \, d\omega \tag{3.7}$$

Then, in an even weaker sense the two last equations can be written as

$$I = \sum_{n=0}^{N_0} |\overline{f_n}\rangle\langle \tilde{f}_n| + \int_0^\infty |\overline{f_\omega}\rangle\langle \tilde{f}_\omega| \, d\omega \tag{3.8}$$

$$H = \sum_{n=0}^{N_0} z|\overline{f_n}\rangle\langle \tilde{f}_n| + \int_0^\infty \omega|\overline{f_\omega}\rangle\langle \tilde{f}_\omega| \, d\omega \tag{3.9}$$

The bases $\{|\overline{f_n}\rangle, |\overline{f_\omega}\rangle\}$ and $\{|\tilde{f}_n\rangle, |\tilde{f}_\omega\rangle\}$ are a biorthonormal system [5, 16], namely

$$\langle \tilde{f}_n|\overline{f_m}\rangle = \delta_{nm}, \qquad \langle \tilde{f}_n|\overline{f_\omega}\rangle = 0, \qquad \langle \tilde{f}_\omega|\overline{f_n}\rangle = 0, \qquad \langle \tilde{f}_\omega|\overline{f_{\omega'}}\rangle = \delta(\omega - \omega') \tag{3.10}$$

Also, it can be proved that

$$\langle \overline{f_n}|\overline{f_m}\rangle = \delta_{nm}\varepsilon_n \tag{3.11}$$

$$\langle \tilde{f}_n|\tilde{f}_m\rangle = \delta_{nm}\varepsilon_n \tag{3.12}$$

where $\varepsilon_n = 1$ if $\operatorname{Im} z_n = 0$ and $\varepsilon_n = 0$ otherwise [15, 12]; namely the states with $\operatorname{Im} z_n \neq 0$ are "ghosts" with vanishing norm. This fact is evident since if $|\overline{f_n}\rangle$ is one of these ghosts, from Eq. (3.1), we have

$$\begin{aligned}\langle \overline{f_n}|\overline{f_n}\rangle &= \langle \overline{f_n}|\left(\sum_{n=0}^{N} |\omega_n\rangle\langle\omega_n| + \int_0^\infty |\omega_+\rangle\langle\omega_+| d\omega\right)|\overline{f_n}\rangle \\ &= \langle \overline{f_n}|\left(\sum_{n=0}^{N} |\tilde{f}_n\rangle\langle \tilde{f}_n| + \int_0^\infty |\tilde{f}_\omega\rangle\langle \tilde{f}_\omega| d\omega\right)|\overline{f_n}\rangle = 0\end{aligned} \tag{3.13}$$

where we have used Eq. (3.8) and that $|\tilde{f}_n\rangle = |\omega_n\rangle$ and $|\tilde{f}_\omega\rangle = |\omega_+\rangle$ [Eq. (44) of ref. 15].

We will sometimes find it useful to write all these equations using a shorthand notation where we will call the basis $\{|\omega_0\rangle, |\omega+\rangle\}$ just $\{|i\rangle\}$, the basis $\{|\bar{f}_n\rangle, |\bar{f}_\omega\rangle\}$ just $\{|\bar{i}\rangle\}$, and the basis $\{|\tilde{f}_n\rangle, |\tilde{f}_\omega\rangle\}$ just $|\{|\tilde{i}\rangle\}$. Then Eq. (3.6) reads

$$|\psi\rangle = \sum_i |\bar{i}\rangle\langle\tilde{i}|\psi\rangle = \sum_i \psi_i|\bar{i}\rangle \tag{3.14}$$

and also we will conventionally say that $\langle\tilde{i}|\psi\rangle \in \mathcal{S}_-$. Equation (3.7) reads

$$H|\psi\rangle = \sum_i z_i|\bar{i}\rangle\langle\tilde{i}|\psi\rangle \tag{3.15}$$

In all these equations $|\bar{i}\rangle \in \phi_+^\times$ and $|\tilde{i}\rangle \in \phi_-^\times$. The biorthonormality of the system $\{|\bar{i}\rangle\}$ and $\{|\tilde{i}\rangle\}$ will be symbolized as

$$\langle\tilde{i}|\bar{j}\rangle = \delta_{ij} \tag{3.16}$$

$$\sum_i |\bar{i}\rangle\langle\tilde{i}| = I \tag{3.17}$$

where the symbols have an obvious meaning [e.g., Eq. (3.17) is a shorthand-notation weak version of Eq.(3.8), etc.].

Also

$$\langle\bar{i}|\bar{j}\rangle = \delta_{ij}\varepsilon_i \tag{3.18}$$

$$\langle\tilde{i}|\tilde{j}\rangle = \delta_{ij}\varepsilon_i \tag{3.19}$$

where $\varepsilon_i = 1$ if Im $z_i = 0$ and $\varepsilon_i = 0$ in all other cases.

4. DENSITY MATRICES

Up to now we have just introduced pure states, but we can rephrase everything using mixed states ρ. In general $\rho \in \Xi \otimes \Xi$, but usually it is considered that it belongs to a Liouville space $\mathcal{L}' = \mathcal{H} \otimes \mathcal{H}$. The time evolution of the mixed states can be obtained by solving the Liouville equation

$$i\frac{d\rho}{dt} = [H, \rho(t)] = L\rho(t) \tag{4.1}$$

where L is the Liouville operator.

From this equation we see that any ρ_* that commutes with H is a stationary state. This state is diagonal in the same basis as H and therefore it can be written as [cf. Eq. (2.5)]

$$\rho_* = \rho_0|\omega_0\rangle\langle\omega_0| + \int_0^\infty \rho_\omega|\omega+\rangle\langle\omega+|\, d\omega \tag{4.2}$$

where we have taken $N = 0$ for simplicity, as we have announced. The second term on the r.h.s. of the last equation implies the existence of a singular structure in the stationary state which was studied at large in ref. 24. So, if we want to develop a rigorous treatment of this singular structure we are forced to consider that $\mathcal{L} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$, where the first $\mathcal{H}$ contains the singular structure and the second factor $\mathcal{H} \otimes \mathcal{H}$ is the usual Liouville space $\mathcal{L}'$, which now will be only considered as a regular structure, and therefore we introduce the following eigenbasis of L:

$$\rho(0) = |\omega_0\rangle\langle\omega_0|, \qquad \rho(0, \omega) = |\omega_0\rangle\langle\omega+|, \qquad \rho(\omega, 0) = |\omega+\rangle\langle\omega_0|$$
$$\beta(\omega) = |\omega+\rangle\langle\omega+|, \qquad \rho(\omega, \omega') = |\omega+\rangle\langle\omega'+| \tag{4.3}$$

this is an orthonormal basis in an inner product that we will define below [cf. Eq. (4.8)].

We can now compute the eigenvalues of the eigenvalues of L:

$$L\rho(0) = 0, \qquad L\rho(0, \omega) = (\omega_0 - \omega)\rho(0, \omega),$$
$$L\rho(\omega, 0) = (\omega - \omega_0)\rho(\omega, 0) \tag{4.4}$$
$$L\beta(\omega) = 0, \qquad L\rho(\omega, \omega') = (\omega - \omega)\rho(\omega, \omega')$$

But for $\rho(\omega, \omega')$ it is better to use Riesz quantum numbers:

$$\sigma = \tfrac{1}{2}(\omega + \omega'), \qquad 0 \leq \sigma < \infty$$
$$\nu = \omega - \omega', \qquad -2\sigma \leq \nu \leq 2\sigma \tag{4.5}$$

So we will write the matrices $\rho(\omega, \omega')$ as

$$\rho(\omega, \omega') = \beta(\sigma, \nu) \tag{4.6}$$

So any $\rho \in \mathcal{L}$ can be written as

$$\rho = \rho_0\rho(0) + \int_0^\infty [\rho_{0\omega}\rho(0, \omega) + \rho_{\omega 0}\rho(\omega, 0) + \rho_\omega\beta(\omega)]\, d\omega$$
$$+ \int_0^\infty d\sigma \int_{-2\sigma}^{2\sigma} d\nu\, \rho_{\sigma\nu}\beta(\sigma, \nu) \tag{4.7}$$

The inner product among these ρ is naturally defined as

$$(\rho|\rho') = \rho_0^*\rho_0' + \int_0^\infty [\rho_{0\omega}^*\rho_{0\omega}' + \rho_{\omega 0}^*\rho_{\omega 0}' + \rho_\omega^*\rho_\omega']\, d\omega$$
$$+ \int_0^\infty d\sigma \int_{-2\sigma}^{2\sigma} d\nu\, \rho_{\sigma\nu}^*\rho_{\sigma\nu}' \tag{4.8}$$

From Eq. (4.4), we have

$$L = \int_0^\infty (\omega_0 - \omega)[\rho(0, \omega)\rho^\dagger(0, \omega) - \rho(\omega, 0)\rho^\dagger(\omega, 0)]\, d\omega + \int_0^\infty d\sigma \int_{-2\sigma}^{2\sigma} \nu\, d\nu\, \beta(\sigma, \nu)\beta^\dagger(\sigma, \nu) \tag{4.9}$$

Now let us make the analytic continuation.

The diagonal elements $\rho(0)$ and $\beta(\omega)$ will remain untouched, since they correspond to the stationary state, but we will require that

$$\rho_\omega \in \mathcal{S} \tag{4.10}$$

The terms $\rho(0, \omega)$ and $\rho(\omega, 0)$ can be treated as in the last section, so they have only one variable ω, so we will ask that

$$\rho_{\omega 0} \in \mathcal{S} \cap H^2_-\big|_{\mathbb{R}+} = \mathcal{S}_-, \qquad \rho_{0\omega} \in \mathcal{S} \cap H^2_+\big|_{\mathbb{R}+} = \mathcal{S}_+ \tag{4.11}$$

Finally, let us consider the term $\beta(\sigma, \nu)$. We could promote both real variables σ and ν to complex variables, but, as ν is the eigenvalue of the Liouville operator, it is only necessary to promote $\nu \to z \in \mathbb{C}$ [17] and leave σ real. Precisely, as

$$\rho_{\sigma\nu} = (\beta(\sigma, \nu)\big|\rho) \tag{4.12}$$

we can consider the complex-valued function of z:

$$\rho_{\sigma z} = (\beta(\sigma, z)\big|\rho) \tag{4.13}$$

and to ask that

$$\rho_{\sigma\nu} \in \mathcal{S} \cap H^2_-\big|_{-2\sigma}^{2\sigma} = \mathcal{S}_-^{(\sigma)} \tag{4.14}$$

for any $\sigma \geq 0$, thus $\rho_{\sigma z}$ will be an analytic function of z in the lower half-plane. Then we will say that $\rho \in \Phi_-$ if Eqs. (4.10), (4.11), and (4.14) are $\rho_{\omega 0} \in \mathcal{S} \cap H^2_-\big|_{\mathbb{R}+} = \mathcal{S}_-$, $\rho_{0\omega} \in \mathcal{S} \cap H^2_+\big|_{\mathbb{R}+} = \mathcal{S}_+$. We also define a space $\mathcal{L}_-$ such that if $\rho \in \mathcal{L}_-$ we simply have

$$\rho_{\omega 0} \in H^2_-\big|_{\mathbb{R}+}, \qquad \rho_{0\omega} \in H^2_+\big|_{\mathbb{R}+}$$

$$\rho_{\sigma\nu} \in H^2_-\big|_{-2\sigma}^{2\sigma}$$

Let us now define the time-inverted spaces Φ_+ and $\mathcal{L}_+$. If Eq. (4.10) is satisfied and

$$\rho_{\omega 0} \in \mathcal{S} \cap H^2_+\big|_{\mathbb{R}_+} = \mathcal{S}_+, \qquad \rho_{0\omega} \in \mathcal{S} \cap H^2_-\big|_{\mathbb{R}_+} = \mathcal{S}_- \tag{4.15}$$

$$\rho_{\sigma\nu} \in \mathcal{S} \cap H^2_+\big|_{-2\sigma}^{2\sigma} = \mathcal{S}_+^{(\sigma)} \tag{4.16}$$

for any $\sigma \geq 0$, then in this case $\rho_{\sigma z}$ will be an analytic function of z in the upper half-plane, and we will say that $\rho \in \Phi_+$ [but in the definition of this space the basis $|\omega+\rangle$ of Eq. (4.3) must be changed by the basis $|\omega-\rangle$].

We also define a space $\mathcal{L}_+$ such that if $\rho \in \mathcal{L}_+$, we have

$$\rho_{\omega 0} \in H^2_+\big|_{\mathbb{R}_+}, \qquad \rho_{0\omega} \in H^2_-\big|_{\mathbb{R}_+}$$

$$\rho_{\sigma\nu} \in H^2_+\big|_{-2\sigma}^{2\sigma}$$

where we have also changed the basis as before.

Let us now consider the poles.

For the terms $\rho(0, \omega)$ and $\rho(\omega, 0)$ we will find those of the last section, and we can repeat the analytic continuation up to the curve Γ of Section 3.

For the terms $\beta(\omega)$ and $\beta(\sigma, \nu)$, for some fixed σ and for every pole z_n of the S-matrix we will find at the ν or z plane two poles $\pm 2(z_n - \sigma)$ (and also a pole at $\nu = z = 0$ coming from the singular structure of the continuous field, the $\beta(\omega)$ term), that we will call ζ_l. Also, it may happen that for some σ_j extra poles ζ_l^j may appear [18]. So introducing a curve C in the lower half-plane that goes, under all the poles, from -2σ to 2σ of the real axis (Fig. 1) and using, as in the pure-states case, the Cauchy theorem, if $\rho \in \Phi_+$ and $\rho' \in \Phi_-$, we obtain that

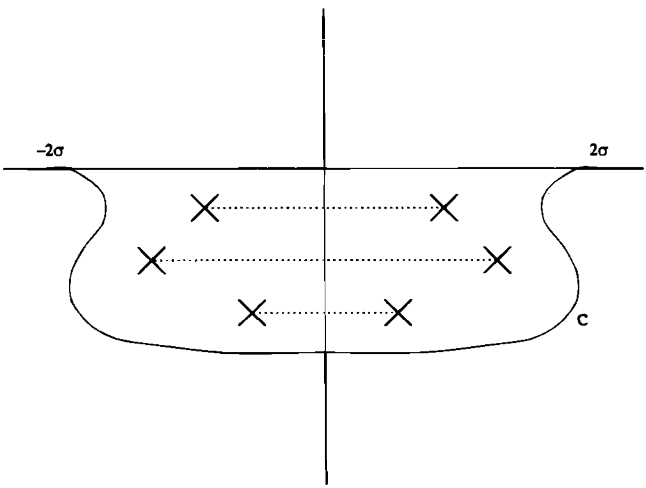

Fig. 1. The poles and the curve C.

$$
\begin{aligned}
(\rho|\rho') = \Bigg(\rho\Bigg|\Bigg\{ & \rho(0)\rho^{\dagger}(0) \\
& + \sum_n [\rho(\overline{z_n, 0})\rho^{\dagger}(\widetilde{z_n, 0}) + \rho(\overline{0, z_n})\rho^{\dagger}(\widetilde{0, z_n})] \\
& + \int_{\Gamma} [\rho(\overline{z, 0})\rho^{\dagger}(\widetilde{z, 0}) + \rho(\overline{0, z})\rho^{\dagger}(\widetilde{0, z})]\, dz \\
& + \sum_{jl} \overline{\beta(\sigma_j, \zeta_l^j)}\, \widetilde{\beta^{\dagger}(\sigma_j, \zeta_l^j)} + \int_0^{\infty} d\sigma \Bigg[\beta(\sigma)\beta^{\dagger}(\sigma) \\
& + \sum_l \beta(\overline{\sigma, \zeta_l})\beta^{\dagger}(\widetilde{\sigma, \zeta_l}) + \int_C \beta(\overline{\sigma z})\beta^{\dagger}(\widetilde{\sigma z})\, dz \Bigg]\Bigg\}\Bigg|\rho'\Bigg)
\end{aligned}
\tag{4.17}
$$

Namely, in the weak sense

$$
\begin{aligned}
I = \rho(0)\rho^{\dagger}(0) & + \sum_n \Bigg[\rho(\overline{z_n, 0})\rho^{\dagger}(\widetilde{z_n, 0}) + \rho(\overline{0, z_n})\rho^{\dagger}(\widetilde{0, z_n}) \\
& + \int_{\Gamma} [\rho(\overline{z, 0})\rho^{\dagger}(\widetilde{z, 0}) + \rho(\overline{0, z})\rho^{\dagger}(\widetilde{0, z})]\, dz \\
& + \sum_{jl} \overline{\beta(\sigma_j, \zeta_l^j)}\, \widetilde{\beta^{\dagger}(\sigma_j, \zeta_l^j)} + \int_0^{\infty} d\sigma \Bigg[\beta(\sigma)\beta^{\dagger}(\sigma) \\
& + \sum_l \beta(\overline{\sigma, \zeta_l})\beta^{\dagger}(\widetilde{\sigma, \zeta_l}) + \int_C \beta(\overline{\sigma z})\beta^{\dagger}(\widetilde{\sigma z})\, dz \Bigg]
\end{aligned}
\tag{4.18}
$$

In these equations the presence of the poles coming from the singular structure (in each σ = const plane) is represented by the terms $\beta(\sigma)\beta^{\dagger}(\sigma)$.

Then we can write any $\rho \in \Phi_-$ as

$$
\begin{aligned}
\rho = {} & \rho_0\rho(0) + \sum_n [\rho_{n0}\rho(\overline{z_n, 0}) + \rho_{0n}\rho(\overline{0, z_n}) + \int_{\Gamma} [\rho_{z0}\rho(\overline{z, 0}) + \rho_{0z}\rho(\overline{0, z})]\, dz \\
& + \sum_{jl} \overline{\rho_{jl}\beta(\sigma_j, \zeta_l^j)} + \int_0^{\infty} d\sigma \Bigg[\rho_{\sigma}\beta(\sigma) + \sum_l \rho_{\sigma l}\beta(\overline{\sigma, \zeta_l}) + \int_C \rho_{\sigma z}\beta(\overline{\sigma z})\, dz \Bigg]
\end{aligned}
\tag{4.19}
$$

Analogously, from the analytic continuation of the Liouville operator we obtain

$$(\rho|L|\rho') = (\rho|\{\sum_n [(z_n - \omega_0)\rho(\overline{z_n, 0})\rho^\dagger(\widetilde{z_n, 0}) + (\omega_0 - z_n^*)\rho(\overline{0, z_n})\rho^\dagger(0, z_n)]$$
$$+ \int_\Gamma [(z - \omega_0)\rho(\overline{z, 0})\rho^\dagger(\widetilde{z, 0}) + (\omega_0 - z^*)\rho(\overline{0, z})\rho^\dagger(\widetilde{0, z})]\, dz$$
$$+ \sum_{jl} \overline{\beta(\sigma_j, \zeta_l^j)}\, \widetilde{\beta^\dagger(\sigma_j, \zeta_l^j)} + \int_0^\infty d\sigma \Bigg[\sum_l \zeta_l \beta(\overline{\sigma, \zeta_l})\beta^\dagger(\widetilde{\sigma, \zeta_l})|\rho')$$
$$+ \int_C z\beta(\overline{\sigma, z})\beta^\dagger(\widetilde{\sigma, z})|\rho')\, dz \Bigg]\Bigg\}|\rho' \Bigg) \tag{4.20}$$

where $\zeta_l = \overline{\nu}_l - i\gamma_l$, $\gamma_l \geq 0$ and $\zeta_l^j = \overline{\nu_l^j} - i\gamma_l^j$, $\gamma_l^j \geq 0$, and as in the pure-states case, Γ is a curve that goes from 0 to $+\infty$ of the real axis under all the poles of the lower half-plane. Namely, in the weak sense, the Liouville operator reads

$$L = \Bigg\{\sum_n [(z_n - \omega_0)\rho(\overline{z_n, 0})\rho^\dagger(\widetilde{z_n, 0}) + (\omega_0 - z_n^*)\rho(\overline{0, z_n})\rho^\dagger(0, z_n)]$$
$$+ \int_\Gamma [(z - \omega_0)\rho(\overline{z, 0})\rho^\dagger(\widetilde{z, 0}) + (\omega_0 - z^*)\rho(\overline{0, z})\rho^\dagger(\widetilde{0, z})]\, dz$$
$$+ \sum_{jl} \zeta_l^j\overline{\beta(\sigma_j, \zeta_l^j)}\, \widetilde{\beta^\dagger(\sigma_j, \zeta_l^j)} + \int_0^\infty d\sigma \Bigg[\sum_l \zeta_l \overline{\beta(\sigma, \zeta_l)}\beta^\dagger(\widetilde{\sigma, \zeta_l})$$
$$+ \int_{-2\sigma}^{2\sigma} \nu\beta(\overline{\sigma, \nu})\beta^\dagger(\widetilde{\sigma, \nu})\, d\nu \Bigg] \tag{4.21}$$

As in the pure-states case the bases

$$\{\rho(0),\ \rho(\overline{0, z_n}),\ \rho(\overline{z_n, 0}),\ \rho(\overline{0, z}),\ \rho(\overline{z, 0}),\ \beta(\omega),\ \overline{\beta(\sigma_j, \zeta_l^j)},\ \beta(\overline{\sigma, \zeta_l}),\ \beta(\overline{\sigma, z})\}$$
$$\{(\rho(0),\ \rho(\widetilde{0, z_n}),\ \rho(\widetilde{z_n, 0}),\ \rho(\widetilde{0, z}),\ \rho(\widetilde{z, 0}),\ \beta(\omega),\ \widetilde{\beta(\sigma_j, \zeta_l^j)},\ \beta(\widetilde{\sigma, \zeta_l}),\ \beta(\widetilde{\sigma, z})\}$$

are a biorthonormal system under the inner product (4.8).[3] Also, as in the pure case,

$$(\beta(\overline{\sigma, \zeta_l}|\beta(\overline{\sigma', \zeta_{l'}})) = \delta_{\sigma\sigma'}\delta_{ll'}\varepsilon_l$$
$$(\beta(\widetilde{\sigma, \zeta_l}|\beta(\tilde{\sigma}', \zeta_{l'})) = \delta_{\sigma\sigma'}\delta_{ll'}\varepsilon_l \tag{4.22}$$

where $\varepsilon_l = 0$ if Im $\zeta_l \neq 0$ and $\varepsilon_l = 1$ if Im $\zeta_l = 0$, so the states corresponding to complex poles are ghosts as before. The same thing happens with $\rho(0, z_n)$, $\rho(z_n, 0)$, $\rho(\widetilde{0, z_n})$, and $\rho(\widetilde{z_n, 0})$.

[3] From now on we will consider that the discrete index σ_j is included in the continuous index σ, and also ζ_l^j is included in ζ_l. Nevertheless we will conserve the terms σ_j in all the spectral decompositions.

We can generalize the definition of trace as

$$Tr\rho = \sum_i \langle i|\rho|i\rangle = \left(\rho\Big|\sum_i |i\rangle\langle i|\right) = \left(\rho\Big|\sum_i \beta(i0)\right) = (\rho|I) \qquad (4.23)$$

where $\{|i\rangle\}$ is any basis of $\mathcal{H}$. Now using the inner product (4.8), it can be easily proved that all the trace of all the off-diagonal terms vanish. Therefore the trace of all the ghosts vanishes.

Finally, equations similar to Eqs. (3.1) of ref. 15 can be obtained, and using these equations it can be proved that, if $\zeta_l, \zeta'_l, \zeta''_l, \ldots$ are complex, then

$$Tr\beta(\overline{\sigma, \zeta_l})\beta(\overline{\sigma', \zeta_{l'}}) = 0, \qquad \beta(\overline{\sigma, \zeta_l})\beta(\overline{\sigma', \zeta_{l'}})\beta(\overline{\sigma'', \zeta_{l''}}) \cdots = 0 \qquad (4.24)$$

This equation follows also from Eq. (12.12) and says that the trace of the product of two ghosts and the product of three or more ghosts vanish.

Since the Liouville space is a Hilbert space $\mathcal{L} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ we will have the Gel'fand triplets

$$\Phi_- \subset \mathcal{L}_- \subset \Phi_-^\times \qquad (4.25)$$

$$\Phi_+ \subset \mathcal{L}_+ \subset \Phi_+^\times \qquad (4.26)$$

Let us observe that, in order to satisfy Eq. (4.3_5) it is sufficient that the regular part of $\rho \in \mathcal{S}_- \otimes \mathcal{S}_+$ [since really in the second factor of Eq. (4.3) there is a bra, not a ket]. Thus, as we will see in more detail in Section 12 [cf. Eq. (12.7)] $\mathcal{S} \oplus (\phi_- \otimes \phi_-) \subset \Phi_-$ and $\mathcal{S} \oplus (\phi_+ \otimes \phi_+) \subset \Phi_+$.

As we will see, Φ_- will be the space of physically admissible states, precisely the space of states such that they evolve with a nondecreasing of entropy according to the second law of thermodynamics. $\Xi \oplus (\Xi \otimes \Xi)\backslash\Phi_-$ is the set of physically nonadmissible states. Φ_+ is the space of states such that they evolve with a nongrowing entropy, and therefore they are clearly nonphysical. Macroscopically the physically admissible evolutions are those that appear in nature, namely those that begin in an unstable state and go toward equilibrium (Gibbs ink drop spreading in a glass of water, a sugar lump dissolving in a cup of coffee, etc.). We will consider that everything is the same in the microscopic case, namely that Φ_- is the space of physically realizable states. The physically nonadmissible evolutions of space Φ_+ can be obtained by the time inversion of the admissible ones, therefore they begin in an equilibrium state and evolve toward an unstable state (the ink or the sugar concentrating spontaneously and creating the drop or the lump). This kind of evolutions does not appear in nature, because the spontaneous appearance of an unstable state by a fluctuation, even if not completely impossible (remember we are developing quantum mechanics, an essentially statistical theory [6]), is highly improbable.

For all these reasons we will consider Φ_- the space of physical states.

5. LINEAR OPERATORS AND OBSERVABLES

We will now consider the linear operators A, which are (anti)linear functional over Φ_- and therefore belong to $\Phi_-^\times$, e.g., a derivative operator belongs to this space.

But these linear operators are merely theoretical or mathematical "observables." Real physical observables are less subtle, e.g., there are not real apparatuses to measure mathematical derivatives. Real physical devices only measure ratios of small but finite quantities. Therefore we can consider that real physical observables live in a space endowed at least with the properties of $\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{S})$. As in the case of the state, it is also useful that these observable have some analytic properties. There are two natural subspaces of $\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{S})$ with definite analytic properties, Φ_- and Φ_+. As we will see, in order to reproduce the relation between the Schrödinger and the Heisenberg pictures in the new theory, we must choose Φ_+ as the space of regular physical observables. As

$$\Phi_-, \Phi_+ \subset \mathcal{L} \tag{5.1}$$

the products between vectors of these two spaces are well defined. Then the mean values of all the observables of Φ_+ in the states of Φ_- are well defined. This property is sufficient to develop a quantum mechanics formalism.

As we will see, in a scattering theory, physical states are related to the preparation, they propagate toward the future in the Schrödinger picture, and they are well represented by states of space Φ_-, while physical observables are related to measurements, they propagate toward the past in the Heisenberg picture, and they are well represented by states of space Φ_+ [8].

After all these considerations the mean value of observable $A \in \Phi_+$ in the states $\rho \in \Phi_-$ is

$$\langle A \rangle_\rho = A[\rho] = (\rho | A) \tag{5.2}$$

where $A[\rho]$ is an (anti)linear functional and

$$\begin{aligned} A = {} & A_0 \rho(0) + \sum_n \left[A_{n0} \rho(\widetilde{z_n, 0}) + A_{0n} \rho(\widetilde{0, z_n}) + \int_\Gamma [A_{z0} \rho(\widetilde{z, 0}) + A_{0z} \rho(\widetilde{0, z})] \, dz \right. \\ & + \sum_{jl} A_{jl} \, \widetilde{\beta(\sigma_j, \zeta_l)} + \int_0^\infty d\sigma \left[A_\sigma \beta(\sigma) + \sum_l A_{\sigma l} \beta(\widetilde{\sigma, \zeta_l}) \right. \\ & \left. + \int_C \rho_{\sigma z} \beta(\widetilde{\sigma, z}) \, dz \right] \end{aligned} \tag{5.3}$$

where the coefficients A must satisfy Eqs. (4.15) and (4.16). To fulfill these conditions, and Eqs. (4.11) and (4.14) for the coefficients of ρ, it is sufficient that the analytic continuation of Eq. (5.2) be possible, one variable in the lower half-plane and the other in the upper half-plane, as we have done in ref. 15.

From the Gel'fand-Maurin theorem [21] we know that we can diagonalize Eq. (5.3) as

$$A = \sum_i a_i |a_i\rangle\langle a_i| \tag{5.4}$$

where in general $a_i \in \mathbb{C}$, i is an index such that the whole spectrum of A is covered by the sum, and $|a_i\rangle$ belongs to some specific rigged Hilbert space. if A is self-adjoint, then obviously $a_i \in \mathbb{R}$ and usually $|a_i\rangle \in \mathcal{S}^\times$. In particular we can expand the energy operator as

$$H = \sum_i h_i |h_i\rangle\langle h_i| \tag{5.5}$$

where $h_i \in \mathbb{R}$ and $|h_i\rangle \in \mathcal{S}^\times$, which can simply be obtained from Eq. (2.4) or Eq. (4.7) as

$$H = \int_0^\infty E|E\rangle\langle E|\, dE = E_0\beta(0) + \int_0^\infty d\sigma\, E_\sigma\beta(\sigma) \tag{5.6}$$

Thus H has two different spectral expansions, both very useful: one as observable Eq. (5.5) and another as an evolution operator equation (3.15), namely, in the shorthand notation of Section 3,

$$H = \sum_i z_i |\bar{i}\rangle\langle\tilde{i}| \tag{5.7}$$

where $z_i \in \mathbb{C}$. The difference between the two expansions comes from the fact that really they are weak equations corresponding to

$$\langle\psi_1|H|\psi_2\rangle = \sum_n h_n \langle\psi_1|h_n\rangle\langle h_n|\psi_2\rangle \tag{5.8}$$

where $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{S}$,

$$\langle\varphi|H|\psi\rangle = \sum_i z_i \langle\varphi|\bar{i}\rangle\langle\tilde{i}|\psi\rangle \tag{5.9}$$

where $|\psi\rangle \in \phi_-$ and $|\varphi\rangle \in \phi_+$.

Now we have all the mathematical objects to formulate our axiomatic theory.

6. AXIOMS

We will follow the main lines of ref. 6. So we postulate:

Axiom 1. To each dynamical variable $\mathcal{R}$ (physical concept) there corresponds a linear operator $R \in \Phi_+ \subset \mathcal{L}_+$ (mathematical object) and the possible values of the dynamical variable are the eigenvalues of the operator.

Axiom 2. To each physical state there corresponds a unique state operator $\rho \in \Phi_- \subset \mathcal{L}_-$. The average value of the dynamical variable $\mathcal{R}$ (e.g., of position, momentum, energy, etc.) represented by the operator R, in the virtual ensemble of events that may result from a preparation procedure for the state, represented by the operator ρ, is

$$\langle \mathcal{R} \rangle_\rho = \frac{Tr[\rho R]}{Tr\rho} \tag{6.1}$$

From these axioms, if we postulate the invariance of the theory under Galilei transformation, the explicit expression of the operators R can be found and Schrödinger and Liouville equations can be deduced as in ref. 6 or 22. Moreover, Planck's constant $\hbar$ appears as a proportionality coefficient between the geometrical generators and the physical magnitudes. In fact, these deductions can be implemented since we have just restricted the domain of definition of the states and of the observables, but all the relevant demonstrations of the quoted references remain valid. We do not reproduce this demonstration here because it is not in our main line of reasoning. So, in order to avoid these demonstrations, even if we maintain the Galilei invariance, we make precise the main features of the time evolution by the following axiom:

Axiom 3. The time evolution of a state $\rho(t) \in \Phi_-$ is

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt} = e^{-iLt}\rho(0) \tag{6.2}$$

where $\rho(0)$, $\rho(t) \in \Phi_-$, and H is the Hamiltonian operator of the system.

In this equation we must use Eq. (5.7) if we want to expand H. The exponents iHt really read $i\hbar^{-1}Ht$, so it is this axiom that introduces the universal constant $\hbar$. Of course, we take $\hbar = 1$ below. From this axiom we also can demonstrate that $\rho(t)$ satisfies the Liouville equation (4.1).

Since we have restricted the spaces of definition of the observables and the states to two spaces which are contained in $\mathcal{L}$, nothing unphysical can really happen. Furthermore, we obviously retain the main result of the usual quantum physics, as we will see in the next two sections, but with the new axiomatic structure we will gain new results that are, in fact, confirmed by experimental evidence.

7. FIRST CONSEQUENCES OF THE AXIOMS

We can now obtain the first consequences of the axioms following ref. 6:

(a) As ρ is both in the numerator and in the denominator of Eq. (6.1), we can normalize the state as

$$Tr\rho = 1 \tag{7.1}$$

(b) If we postulate that projectors like $P = |u\rangle\langle u|$ are observables, and since their eigenvalues are 0 and 1, we can see that $\langle P\rangle_\rho = \langle u|\rho|u\rangle$ is real and positive, so (cf. Theorem 1, ref. 6)

$$\rho = \rho^\dagger, \qquad \langle u|\rho|u\rangle \geq 0 \tag{7.2}$$

(c) As $\rho = \rho^\dagger$, ρ can be expanded as

$$\rho = \sum_i \rho_i |\rho_i\rangle\langle\rho_i| \tag{7.3}$$

where $\rho_i \in \mathbb{R}$, and $|\rho_i\rangle$ belongs to some adequate rigged Hilbert space. Then from the three first equations of this section we can obtain that

$$\sum_i \rho_i = 1, \qquad \rho_i = \rho_i^*, \qquad 0 \leq \rho_i \leq 1 \tag{7.4}$$

(d) If we postulate that the mean value of any dynamical variable must be real, i.e.,

$$\langle\mathcal{R}\rangle_\rho \in \mathbb{R} \tag{7.5}$$

then, for any pure state $\rho = |v\rangle\langle v|$, $|v\rangle \in \phi_-$, we have

$$\langle\mathcal{R}\rangle_\rho = \langle v|R|v\rangle \in \mathbb{R} \tag{7.6}$$

so, according to Theorem 1 of ref. 6, we have

$$R = R^\dagger \tag{7.7}$$

so we can expand R as in Eq. (4.9), namely

$$R = \sum_n r_n |r_n\rangle\langle r_n| \tag{7.8}$$

where $r_n \in \mathbb{R}$, and $|r_n\rangle$ belongs to some adequate rigged Hilbert space.

8. PROBABILITIES

From Axiom 1 we have that

$$\langle\mathcal{R}\rangle_\rho = \sum_n r_n p_n(\rho) \tag{8.1}$$

where $p_n(\rho)$ is the probability to obtain the measurement r_n when we measure the dynamical variable $\mathcal{R}$ in the quantum state ρ. From Axiom 2 we also have

$$\langle\mathcal{R}\rangle_\rho = R[\rho] = Tr\left[\left(\sum_n r_n |r_n\rangle\langle r_n|\right)\rho\right] = \sum_n r_n \langle r_n|\rho|r_n\rangle \tag{8.2}$$

In order for the last two equations to be equal it is sufficient that

$$p_n(\rho) = \langle r_n | \rho | r_n \rangle \tag{8.3}$$

It can be proved that this condition is also necessary if we repeat the corresponding demonstration of ref. 6.

Then, for every state ρ and every complete set of commuting observables $\{R^{(\alpha)}\}$, we can compute the probability to obtain the measurement $r_n^{(\alpha)}$ for the observable $R^{(\alpha)}$. In fact, we can expand the observable as

$$R^{(\alpha)} = \sum_n r_n^{(\alpha)} | r_n^{(\alpha)} \rangle \langle r_n^{(\alpha)} | \tag{8.4}$$

and the probability is

$$p_n^{(\alpha)}(\rho) = \langle r_n^{(\alpha)} | \rho | r_n^{(\alpha)} \rangle \tag{8.5}$$

So we can see that ρ really defines the *quantum state* of the system since, knowing ρ, we can obtain the probability of any measurement for any observable of the complete set of commuting observables. This is in fact the maximal information that we can obtain from a quantum state ρ, and in consequence this information also defines the quantum state of the system.

9. TIME ASYMMETRY AND IRREVERSIBILITY

In the last two sections we briefly reviewed some results of ordinary quantum mechanics that turn out to be also valid in the new theory. It would be quite boring to continue this road reobtaining well-known results, so we will now consider the new features.

We will say that:

Time asymmetry is the property of some single objects that turn out to be asymmetric under the action of the time-inversion Wigner operator K, e.g., nonreal states $|\psi\rangle$, defined as the states such that $K|\psi\rangle \neq |\psi\rangle$. In our case these objects are always statistical objects from the ensemble we are considering, since we are developing a statistical theory. Therefore the time asymmetry of the particular evolution of the members of the ensemble will be never taken into account.

Non-time-reversal invariance is the property of some set of objects which are not invariant under K, e.g., the space ϕ_- which has the property (2.9).

Irreversibility is the property of some physical time evolutions such that the time-inverted evolution turns out to be nonphysical, namely it is physically forbidden [1, 2].

To make the term more precise, the irreversibility just introduced is the *dynamical irreversibility* (as we will see in a moment, this irreversibility stems directly from the axioms). *Thermodynamic irreversibility* will be defined in

Section 13 as the increase of entropy (and we will see that more elements must be added to define this notion).

From the just-quoted Eq. (2.9) and the definitions at the beginning of Section 4 we have that

$$\mathcal{K}: \quad \Phi_- \to \Phi_+ \neq \Phi_- \tag{9.1}$$

where $\mathcal{K}\rho = K\rho K^\dagger$, and we can see that the physically admissible quantum state space of the theory is not time-reversal invariant.

From ref. 15, Eq. (4.2), and Eq. (4.3), we know that

$$e^{-iHt}: \quad \phi_- \to \phi_- \quad \text{if} \quad t > 0, \qquad e^{-iHt}: \quad \phi_+ \to \phi_+ \quad \text{if} \quad t < 0 \tag{9.2}$$

Therefore, using the same demonstration regarding now the analytic properties of the functions of variable ν, it can be proved that

$$e^{-iLt}: \quad \phi_- \to \phi_- \quad \text{if} \quad t > 0, \qquad e^{-iLt}: \quad \Phi_+ \to \Phi_+ \quad \text{if} \quad t < 0 \tag{9.3}$$

so Axiom 3 states that if $\rho(t) \in \Phi_-$, its evolution is only defined for $t > 0$, and therefore the evolution operator e^{-iLt} cannot be physically inverted since its mathematical inverted operator e^{iLt} corresponds to $t < 0$ and therefore it is not well defined within space Φ_-. Namely the inverted evolution is forbidden by Axiom 3. Therefore we have found that the new theory contains dynamical irreversible evolutions.

Of course $t = 0$ is an arbitrary time, so the condition $t > 0$ physically simply means that operators e^{-iHt} and e^{-iLt} are not well defined for $t \to -\infty$ for the state of space Φ_-. Analogously, the condition $t < 0$ means that the same operators are not well defined for $t \to +\infty$ for the states of Φ_+.

10. SCHRÖDINGER AND HEISENBERG PICTURES AND SCATTERING EXPERIMENTS

In the Schrödinger picture, if $\rho(t)$ is the time-variable state of the system and $\mathcal{R}$ is a fixed dynamical variable, from Axioms 2 and 3 we have

$$\langle \mathcal{R} \rangle_\rho(t) = Tr[\rho(t)R] = Tr[e^{-iLt}\rho(0)R] = Tr[e^{-iHt}\rho(0)e^{iHt}R] \tag{10.1}$$

According to Axiom 3, $\rho(t) \in \Phi_-$, so, from Eq. (9.3_1), we know that the last equation is only valid if $t > 0$. Now from the cyclic property of the trace we also have that

$$\langle \mathcal{R} \rangle_\rho(t) = Tr[\rho(0)e^{iHt}Re^{-iHt}] \tag{10.2}$$

so we can define a time-variable Heisenberg operator:

$$R_H(t) = e^{-iH(-t)}Re^{iH(-t)} = e^{-iL(-t)}R \tag{10.3}$$

Then we have the Heisenberg-picture equation

$$\langle \mathcal{R} \rangle_{\rho}(t) = Tr[\rho(0) R_H(t)] \tag{10.4}$$

But from Eq. (9.3_2) and since $-t < 0$, we know that the last time equation is only valid if $R \in \Phi_+$. This fact justifies both the choice of the operator space done in Axiom 1 and what we said in Section 4.

In other words, in a scattering experiment [8, 11] the states are prepared at a time t_1 and propagate toward the future and therefore to times $t > t_1$, so according to Eq. (9.3_1), $\rho \in \Phi_-$. At time $t_2 > t_1$ dynamical variables $\mathcal{R}$ are measured, i.e., the S-matrix and the corresponding probabilities are obtained. But we can invert the procedure and propagate the dynamical variables $\mathcal{R}$ and the corresponding operators R toward the past down to time $t_1 < t_2$. Then we must propagate R toward the past, therefore according to Eq. (9.3_2), $R \in \Phi_+$. So now we see the motivation of the choice of the spaces for ρ and R made in Axioms 1 and 2, namely $\rho \in \Phi_-$ and $R \in \Phi_+$.

11. EQUILIBRIUM AND DECOHERENCE

We will approach the problem of equilibrium in four steps: in the first one we will obtain a strong limit, in the second one a weak limit, in the third one the dominant time evolution components, and in the fourth one we will combine the last two to obtain some physical conclusions.

(a) From Eqs. (4.19) and (4.21) we can deduce that if $\rho(t) \in \Phi_-$

$$\begin{aligned} \rho(t) = {} & \rho_0 \rho(0) + \sum_n [\rho_{n0} e^{-i(z_n - \omega_0)t} \rho(\overline{z_n, 0}) + \rho_{0n} e^{-i(\omega_0 - z_n^*)t} \rho(\overline{0, z_n})] \\ & + \int_\Gamma [\rho_{z0} e^{-i(z-\omega_0)t} \rho(\overline{z, 0}) + \rho_{0z} e^{-i(\omega_0 - z)t} \rho(\overline{0, z})]\, dz \\ & + \sum_{jl} \rho_{jl} e^{-i\zeta_l^j t} \beta(\overline{\sigma_j, \zeta_l^j}) + \int_0^\infty d\sigma \left[\rho_\sigma \beta(\sigma) + \sum_l \rho_{\sigma l} e^{-i\zeta_l t} \beta(\overline{\sigma, \zeta_l}) \right. \\ & \left. + \int_C \rho_{\sigma z} e^{-izt} \beta(\overline{\sigma z})\, dz \right] \end{aligned} \tag{11.1}$$

where ζ_l and z_n symbolize the complex poles. If we call, as is traditional,

$$\begin{aligned} z_n &= \overline{\omega}_n - \frac{i}{2} \gamma_n, \qquad \gamma_n > 0 \\ \zeta_l &= \overline{\nu}_l - i\Gamma_l, \qquad \Gamma_l > 0 \end{aligned} \tag{11.2}$$

we have that

$$\rho(t) = \rho_0\rho(0) + \sum_n [\rho_{n0}e^{-i(\overline{\omega}_n-\omega_0)t}e^{-(1/2)\gamma_n t}\rho(\overline{z_n,0}) + \rho_{0n}e^{-i(\omega_0-\overline{\omega}_n)t}e^{-(1/2)\gamma_n t}\rho(\overline{0,z_n})] + \int_\Gamma [\rho_{z0}e^{-i(z-\omega_0)t}\rho(\overline{z,0}) + \rho_{0z}e^{-i(\omega_0-z)t}\rho(\overline{0,z})]\,dz + \sum_{jl}\rho_{jl}e^{-i\overline{\nu}_l^j t}e^{-\Gamma_l^j t}\overline{\beta(\sigma_j,\zeta_l^j)} + \int_0^\infty d\sigma\left[\rho_\sigma\beta(\sigma) + \sum_l \rho_{\sigma l}e^{-i\overline{\nu}_l t}e^{-\Gamma_l t}\beta(\overline{\sigma,\zeta_l}) + \int_C \rho_{\sigma z}e^{-izt}\beta(\overline{\sigma z})\,dz\right] \tag{11.3}$$

For complex poles we have γ_n, $\Gamma_l > 0$, so terms containing these positive gammas vanish when $t \to +\infty$. The terms corresponding the continuous spectra do not vanish since the curves Γ and C can be taken to be contained in the real axis almost anywhere. Then we obtain the strong limit

$$\rho(t) \to \rho_*(t) = \rho_0\rho(0) + \int_\Gamma [\rho_{z0}e^{-i(z-\omega_0)t}\rho(\overline{z,\,0}) + \rho_{0z}e^{-i(\omega_0-z)t}\rho(\overline{0,\,z})]\,dz + \int_0^\infty d\sigma\left[\rho_\sigma\beta(\sigma) + \int_C \rho_{\sigma z}e^{-izt}\beta(\overline{\sigma z})\,dz\right] \tag{11.4}$$

so any state goes to a state of "dynamical equilibrium." We use the adjective "dynamical" since it is a final state that it is a function of time. If we take into account normalization (7.1), we have

$$Tr\rho(t) = Tr\rho_*(t) = 1 \tag{11.5}$$

so

$$\rho_0 + \int_0^\infty \rho_\sigma\,d\sigma = 1 \tag{11.6}$$

This is certainly an equation that the ρ_0 and ρ_σ must satisfy, but in principle this is the only condition.

So in general the dynamical equilibrium state is not unique and depends on the initial conditions. This would be the general case.

(b) Moreover, if we consider that really the ρ are functionals over the observables A, since only the mean values $\langle A\rangle_\rho = A[\rho]$ are physically observed, and we use the Riemann–Lebesgue theorem as in ref. 24, we obtain in a weak sense that

$$\rho(t) \to \rho_* = \rho_0 \rho(0) + \int_0^\infty \rho_\sigma \rho(\sigma)\, d\sigma \tag{11.7}$$

Therefore only the terms on the diagonal remain and we obtain a stationary final equilibrium as a weak limit. Only the state $\rho(0)$, corresponding to the ground state of the discrete spectrum (e.g., an oscillator), and the states $\rho(\sigma)$, corresponding to the diagonal states of the continuous one (e.g., the bath), remain in equilibrium. Thus we have proved that quantum decoherence takes place in our theory.

(c) But, using this method, we just obtain the limit, but we cannot see how this limit is attained. So we will use another approach. We know that the dominant component of the evolution toward equilibrium is given by the pole terms [16]. In fact, this component is an excellent approximation for intermediate times: not too short times, so the Zeno effect would not be important; not too long times, so the Khalfin effect would not be important. Furthermore, experimentally we know that this intermediate period turns out to be very long, since the Khalfin effect has not yet been detected. So if we want to have a very good approximation of the evolution toward equilibrium, we can neglect the regular background field terms of curves Γ and C and only consider the pole terms and the singular diagonal terms; i.e.,

$$\begin{aligned}\rho(t) = \rho_0\rho(0) &+ \sum_n [\rho_{n0} e^{-i(\bar{\omega}_n-\omega_0)t} e^{-(1/2)\gamma_n t} \overline{\rho(z_n, 0)} \\ &+ \rho_{0n} e^{-i(\omega_0-\bar{\omega}_n)t} e^{-(1/2)\gamma_n t} \overline{\rho(0, z_n)}] \\ &+ \sum_{jl} \rho_{jl} e^{-i\bar{\nu}_l^j t} e^{-\Gamma_l^j t} \overline{\beta(\sigma_j, \zeta_l^j)} \\ &+ \int_0^\infty d\sigma \left[\rho_\sigma \beta(\sigma) + \sum_l \rho_{\sigma l} e^{-i\bar{\nu}_l t} e^{-\Gamma_l t} \overline{\beta(\sigma, \zeta_l)}\right]\end{aligned} \tag{11.8}$$

We will write

$$\rho_* = \rho_0\rho(0) + \int_0^\infty \rho_\sigma \rho(\sigma)\, d\sigma \tag{11.9}$$

$$\begin{aligned}e^{-\gamma_{\min} t}\rho_1(t) = \sum_n \Bigg[&\rho_{n0} e^{-i(\bar{\omega}_n-\omega_0)t} e^{-(i/2)\gamma_n t} \overline{\rho(z_n, 0)} \\ &+ \rho_{0n} e^{-i(\omega_0-\bar{\omega}_n)t} e^{-(i/2)\gamma_n t} \overline{\rho(0, z_n)} \\ &+ \sum_{jl} \rho_{jl} e^{-i\bar{\nu}_l^j t} e^{-\Gamma_l^j t} \overline{\beta(\sigma_j, \zeta_l^j)} + \int_0^\infty d\sigma \sum_l \rho_{\sigma l} e^{-i\bar{\nu}_l t} e^{-i\Gamma_l} \overline{\beta(\sigma, \zeta_l)}\Bigg]\end{aligned} \tag{11.10}$$

where we have made explicit the minimum of the γ and the Γ in the l.h.s. of the second equation. Since $\gamma, \Gamma > 0$, when $t \to \infty$ we have

$$\rho(t) \to \rho_* \tag{11.11}$$

So in this case we obtain the usual stationary equilibrium, which is not time dependent (but it still depends on the initial condition, through the ρ_0 and the ρ_σ; we will find an equilibrium independent of these conditions in Section 14). The normalization condition is still Eq. (11.6) and we have

$$Tr\rho_1(t) = 0 \tag{11.12}$$

which is also a consequence of the fact that the ghost has vanishing trace.

So again we have obtained the usual equilibrium state and, as the off-diagonal terms vanish when $t \to \infty$, the phenomenon of decoherence is also proved.

The present method has been used to study decoherence in the cosmological case [25].

12. CONSERVATION OF THE NORM, THE TRACE, AND THE ENERGY. LYAPUNOV VARIABLES. SURVIVAL PROBABILITY

(a) In Eq. (11.3) we have shown that some states of the theory vanish for $t \to +\infty$, precisely the "ghost" states such that $\gamma_n, \Gamma_l > 0$. Then one might wonder if the trace, the norm, and the energy are conserved in our theory. In fact, this is so, since we know that the trace of the off-diagonal terms vanishes, so we have

$$Tr\rho(t) = \rho_0 + \int_0^\infty \rho_\sigma \, d\sigma = 1 \tag{12.1}$$

Also

$$Tr[\rho(t)H] = \omega_0\rho_0 + \int_0^\infty \sigma\rho_\sigma \, d\sigma = \text{const} \tag{12.2}$$

so the trace and the mean value of the energy are constant.

In the pure state case these equations read

$$\langle\psi|\psi\rangle = 1 \tag{12.3}$$

$$\langle\psi|H|\psi\rangle = \text{const} \tag{12.4}$$

so the norm of pure states $|\psi\rangle$ is also a constant.

(b) To continue, let us repeat the reasoning at the beginning of this section in a different case: we will use another basis (precisely, the one that can be obtained by the products of the basis of Section 3) and the shorthand notation of Section 3 and let $N \neq 0$. Let us consider the space $\mathcal{S} \oplus (\phi_- \otimes \phi_-)$ and a state $\rho \in \mathcal{S} \oplus (\phi_- \otimes \phi_-)$ which we can develop as

$$\rho = \sum_i \rho_i |i\rangle\langle i| + \sum_{ij} \rho_{ij} |\bar{i}\rangle\langle \bar{j}| \tag{12.5}$$

where $\rho_i \in \mathcal{S}$, $\rho_{ij} \in \mathcal{S}_- \otimes \mathcal{S}_+$. Now as

$$\rho_{ij} = \rho_{\sigma+(1/2)\nu,\sigma-(1/2)\nu} = \rho_{\rho\nu} \tag{12.6}$$

it turns out that $\rho \in \Phi_-$, since $\rho_{\sigma n}$ as a function of ν belongs to $\mathcal{S}_-^{(\sigma)}$; then we can conclude that

$$\mathcal{S} \oplus (\phi_- \otimes \phi_-) \subset \Phi_- \subset \mathcal{L}_- \subset \Phi_-^\times \subset \mathcal{S}^\times \oplus (\phi_-^\times \otimes \phi_-^\times) \tag{12.7}$$

So any function $\beta \in \Phi_-^\times$ [let say $\rho \in \Phi_-$ or $\overline{\beta(\sigma, \xi_l)} \in \Phi_-^\times$] can also be expanded as in Eq. (12.5) and then

$$\begin{aligned} \beta(t) &= e^{-iHt}\left(\sum_i \beta_i |i\rangle\langle i| + \sum_{ij} \beta_{ij} |\bar{i}\rangle\langle \bar{j}| \right) e^{iHt} \\ &= \sum_i \beta_i |i\rangle\langle i| + \sum_{ij} \beta_{ij} e^{-i(z_i^* - z_j^*)} |\bar{i}\rangle\langle \bar{j}| \end{aligned} \tag{12.8}$$

Then as usual, if we call

$$z_i = \omega_i - \frac{i}{2}\gamma_i, \qquad \gamma_i > 0 \tag{12.9}$$

we have

$$\begin{aligned} \beta(t) &= \sum_i \beta_i |i\rangle\langle i| + \sum_{ij} \beta_{ij} e^{-i(\omega_j - \omega_i)t} e^{-(1/2)(\gamma_i + \gamma_j)} |\bar{i}\rangle\langle \bar{j}| \\ &= \sum_i \beta_i |i\rangle\langle i| + \sum_{IJ} \beta_{IJ} e^{-i(\omega_J - \omega_I)t} |\omega_I\rangle\langle \omega_J| \\ &\quad + \sum_{ij \neq IJ} \beta_{ij} e^{-i(\omega_j - \omega_i)t} e^{-(1/2)(\gamma_i + \gamma_j)t} |\bar{i}\rangle\langle \bar{j}| \end{aligned} \tag{12.10}$$

where again the indices I, J, . . . correspond to the real poles of the real continuous spectrum ($ij \neq IJ$ means that either i or j or both correspond to "ghost" states) and as $\gamma_i + \gamma_j > 0$, so if we take $t \to \infty$, we can see that we can expand $\beta_*(t)$ and $e^{-\gamma_{\min} t}\beta_1(t)$ as

$$\beta_*(t) = \sum_i \beta_i |i\rangle\langle i| + \sum_{IJ} \beta_{IJ} e^{-i(\omega_J - \omega_I)t} |\omega_I\rangle\langle \omega_J| \tag{12.11}$$

$$e^{-\gamma_{\min} t}\beta_1(t) = \sum_{ij \neq IJ} \beta_{ij} e^{-i(\omega_j - \omega_i)t} e^{-(1/2)(\gamma_i + \gamma_j)t} |\bar{i}\rangle\langle \bar{j}| \tag{12.12}$$

Finally on this basis the conservation of the norm and the energy read

$$Tr\beta(t) = \sum_i \beta_i = 1 \tag{12.13}$$

$$Tr(\beta(t)H) = \sum_i \omega_i\beta_i = \text{const} \tag{12.14}$$

so we have proved, in another basis, that the trace and the mean value of the energy are constant.

Up to now, every scalar we have introduced is time constant and it seems impossible to define Lyapunov variables. But again let us try to find these results, using now another approach, considering only the regular component $\rho_{reg} \in \mathcal{H} \otimes \mathcal{H}$ of ρ and using the restriction of the trace on the regular component, namely

$$tr\rho_{reg} = \sum_i \langle i|\rho_{reg}|i\rangle = \sum_i \langle \tilde{i}|\rho_{reg}|\bar{i}\rangle \tag{12.15}$$

where the last equation can be obtained by analytic continuation. Then we have

$$tr\rho_{reg}(t) = \sum_i \rho_{ii}e^{-\gamma_i t}\langle \bar{i}|\bar{i}\rangle = \sum_i \rho_{ii}e^{-i\gamma_i t}\varepsilon_i = \sum_I \rho_{II} = \text{const} \tag{12.16}$$

$$\begin{aligned} tr[\rho_{reg}(t)H] &= \sum_i \rho_{ii}\left(\overline{\omega}_i - \frac{i}{2}\gamma_i\right)e^{-\gamma_i t}\langle \bar{i}|\bar{i}\rangle \\ &= \sum_i \rho_{ii}\left(\overline{\omega}_i - \frac{i}{2}\gamma_i\right)e^{-i\gamma_i t}\varepsilon_i = \sum_I \omega_I\rho_{II} = \text{const} \end{aligned} \tag{12.17}$$

so again everything is constant ad we cannot find Lyapunov variables.

From Eq. (12.12) we can now prove Eq. (4.24). In fact, in $\rho_1(t)$ or $\beta_1(t)$ there are only terms like $|\text{no-ghost}\rangle\langle\text{ghost}|$, $|\text{ghost}\rangle\langle\text{no-ghost}|$, and $|\text{ghost}\rangle\langle\text{ghost}|$. If we compute $\rho_1^2(t)$ or $\beta_1^2(t)$ only the combination $|\text{ghost}\rangle\langle\text{no-ghost}|\text{no-ghost}\rangle\langle\text{ghost}|$ survives, so these squares have only $|\text{ghost}\rangle\langle\text{ghost}|$ terms and their traces vanish. Similarly we can prove that the powers higher than two of $\rho_1(t)$ or $\beta_1(t)$ vanish.

(c) We have just learnt that the usual constants of quantum mechanics remain constant in the new theory, so they are not Lyapunov variables, namely variable, never-decreasing quantities. Nevertheless we can define Lyapunov variables in the new theory if we introduce the new "metric" operators

$$\tilde{M} = \sum_i |\tilde{i}\rangle\langle\tilde{i}|, \qquad \overline{M} = \sum_i |\bar{i}\rangle\langle\bar{i}| \tag{12.18}$$

The role of these operators is to exchange bases $\{|\tilde{i}\rangle\}$ and $\{|\bar{i}\rangle\}$ since from Eq. (3.16) we have

$$\tilde{M}|\bar{i}\rangle = \sum_j |\tilde{j}\rangle\langle\tilde{j}|\bar{i}\rangle = |\tilde{i}\rangle \tag{12.19}$$

$$\overline{M}|\tilde{i} = \sum_j |\bar{j}\rangle\langle\bar{j}|\tilde{i}\rangle = |\bar{i}\rangle \tag{12.20}$$

this would happen, e.g., in the $\beta(t)$ of Eq. (12.8). Therefore

$$\tilde{M}: \quad \phi_- \to \phi_+, \qquad \overline{M}: \quad \phi_+ \to \phi_- \tag{12.21}$$

Then from Eq. (3.16) we have

$$\overline{M}\tilde{M} = \tilde{M}\overline{M} = 1 \tag{12.22}$$

$$\langle\tilde{i}|\overline{M}|\tilde{j}\rangle = \delta_{ij} \tag{12.23}$$

$$\langle\bar{i}|\tilde{M}|\bar{j}\rangle = \delta_{ij} \tag{12.24}$$

so the role of the operators M and $\tilde{M}$ is to eliminate the "ghost" factor ε_i from Eqs. (3.18) and (3.19) and, as we can see, to make a variable what was previously a constant.

Then from Eq. (11.3) we now have

$$\begin{aligned} tr[\rho_{reg}(t)\tilde{M}] &= \sum_i \rho_{ii}e^{-\gamma_i t}\langle\bar{i}|\tilde{M}|\bar{i}\rangle = \sum_I \rho_{II} + \sum_{i\neq I} \rho_{ii}e^{-\gamma_i t} \\ &= \text{const} + \sum_{i\neq I} \rho_{ii}e^{-\gamma_i t} = -Y(t) \end{aligned} \tag{12.25}$$

where now $Y(t)$ is a Lyapunov variable, if, e.g., $\rho_{ii} \geq 0$, and we have

$$Y(t) = \sum_{i\neq I} \rho_{ii}\gamma_i e^{-\gamma_i t} > 0 \tag{12.26}$$

because from Eq. (11.2) $\gamma_i > 0$ if $i \neq I$.

If we want to find a Lyapunov variable without the condition $\rho_{ii} \geq 0$ we can introduce the "linear entropy" [26]

$$tr[(\rho_{reg}(t)\tilde{M})^\dagger \rho_{reg}(t)\tilde{M}] = \sum_I |\rho_{II}|^2 + \sum_{i\neq I} |\rho_{ii}|^2 e^{-\gamma_i t} = -Y_G(t) \tag{12.27}$$

and in this case we surely have $Y_G(t) > 0$ since $|\rho_{ii}|^2 > 0$.

(d) We can also compute survival probabilities and find that in the theory we have nonregular states with pure exponential survival probability. In fact let

$$\rho(t) = |\overline{i(t)}\rangle\langle\overline{i(t)}| \tag{12.28}$$

$|\overline{i(t)}\rangle \in \phi_+^\times$. At $t = 0$ we can consider the observable $R = |\psi\rangle\langle\psi| \in \Phi_+$ and see how the probability $p_i(t)$ to measure the state $\rho(t)$ in the eigenstate $|\psi\rangle$ evolves: Using Eq. (8.3), we find

$$p_i(t) = \langle\psi|\overline{i(t)}\rangle\langle\overline{i(t)}|\psi\rangle = e^{-\gamma_i t}\langle\psi|\overline{i(0)}\rangle\langle\overline{i(0)}|\psi\rangle \tag{12.29}$$

which is an exponentially decaying survival probability with mean life γ_i^{-1}. This life-time turns out to be infinite for the non-ghost states $|\omega_I\rangle$, since in this case $\gamma_I = 0$. Then the non-ghost states are stable states and the ghost are unstable decaying states. The physical meaning of the exponents γ_i is now clear, they are the inverse of the mean-life of the decaying processes within the theory.

In the case of the generic state of Eq. (12.10) and a generic operator $R = \Sigma_\alpha \psi_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ we have

$$\begin{aligned} p_\rho(t) &= \sum_{ij\alpha} \rho_{ij}\psi_\alpha e^{-i(\omega_j-\omega_i)} e^{-(1/2)(\gamma_i+\gamma_j)t} \langle\psi_\alpha|\bar{i}\rangle\langle\bar{j}|\psi_\alpha\rangle \\ &= \sum_{IJ\alpha} \rho_{IJ}\psi_\alpha\langle\psi_\alpha|I\rangle\langle J|\psi_\alpha\rangle e^{-i(\omega_J-\omega_I)t} \\ &\quad + \sum_{ij\neq I} \rho_{ij}\psi_\alpha e^{-i(\omega_j-\omega_i)t} e^{-(1/2)(\gamma_i+\gamma_j)t} \langle\psi_\alpha|\bar{i}\rangle\langle\bar{j}|\psi_\alpha\rangle \end{aligned} \tag{12.30}$$

The second term on the r.h.s. is clearly the dominant exponential component of the survival probability, while the first one gives rise to the Zeno and Khalfin effects, as can be seen in examples [27].

13. ENTROPY

As we have seen, the irreversibility of the time evolution of the new theory allows us to introduce Lyapunov variables. In particular, the equilibrium state $\rho_*(t)$ defined in Section 11 can be used to define a very important Lyapunov variable: a quantum conditional entropy [20]. This entropy coincides with the usual conditional entropy in the classical limit, which has a remarkable property: it never decreases under a generic evolution driven by a Markov operator. It is therefore an excellent candidate for the phenomenological internal entropy. In this paper the quantum conditional entropy is just the quantum analogue of the classical conditional entropy, a extensive, ever-growing Lyapunov variable that vanishes at equilibrium. A rigorous and complete study that relates this entropy with other kinds of entropy is lacking (although see refs. 2, 28, and 29).

The naive definition of quantum conditional entropy of the state ρ would be

$$S = -Tr[\rho \log(\rho\rho_*^{-1})] \tag{13.1}$$

where ρ_* is an equilibrium state like the one of Eq. (11.9). Of course, at equilibrium we have $S = 0$.

In this section we will use the time evolution based on the dominant pole component of Eq. (11.8) (so $N = 0$):

$$
\begin{aligned}
\rho(t) &= \rho_0\rho(0) + \sum_n [\rho_{n0} e^{-i(\overline{\omega}_n-\omega_0)t} e^{-(1/2)\gamma_n t}\rho(\overline{z_n, 0}) \\
&\quad + \rho_{0n} e^{-i(\omega_0-\overline{\omega}_n)t} e^{-(1/2)\gamma_n t}\rho(\overline{0, z_n}) \\
&\quad + \sum_{jl} \rho_{jl} e^{-i\overline{\nu}^j_l t} e^{-\Gamma^j_l t}\overline{\beta(\sigma_j, \zeta^j_l)} \\
&\quad + \int_0^\infty d\sigma \left[\rho_\sigma\beta(\sigma) + \sum_l \rho_{\sigma l} e^{-i\overline{\nu}_l t} e^{-\Gamma_l t}\beta(\overline{\sigma, \zeta_l}) \right] \\
&= \rho_* + e^{-\gamma_{\min} t}\rho_1(t)
\end{aligned} \tag{13.2}
$$

where in the second term on the r.h.s. we have made explicit the slowest damping factor so $\gamma_{\min} = \min(\gamma_n, \Gamma_l) > 0$. all other factors contained in Eq. (13.2) are oscillatory and they have a faster decrease than the slowest damping factor. Since the trace of the off-diagonal vanishes, and using Eq. (4.24), we know that

$$
Tr\rho_1(t) = 0, \qquad Tr\rho_1^2(t) = 0, \qquad \rho_1^n(t) = 0, \qquad \text{if} \quad n > 2 \tag{13.3}
$$

We can also introduce a diagonal factor ρ_*^{-1} among the factors $\rho_1(t)$ and the result will be the same; since these matrices ρ_*^{-1} are diagonal, they only modify the coefficient in the expansion of $\rho_1(t)$, while Eq. (4.24) remains valid. Introducing Eq. (13.2) in Eq. (13.1), we have

$$
S(t) = -Tr[(\rho_*(t) + e^{-\gamma_{\min} t}\rho_1(t))\log(1 + e^{-\gamma_{\min} t}\rho_1(t)\rho_*^{-1}(t))] \tag{13.4}
$$

and expanding the logarithm, we obtain

$$
S(t) = -\tfrac{1}{2} e^{-2\gamma_{\min} t} Tr[\rho_1^2(t)\rho_*^{-1}(t)] + \cdots \tag{13.5}
$$

where the dots symbolize terms with higher powers of $\rho_1(t)$. Then from Eq. (13.3) we can conclude that $S(t) = 0$, namely that this naive entropy is constant and it always coincides with the equilibrium entropy. This result is analogous to those obtained in Eq. (12.1) or Eq. (12.2). So we can conclude, as in the previous section, that if we do not use the metric operators $\overline{M}$ and $\tilde{M}$, or eventually some projector P, we will always obtain constant naive quantities. Thus we must somehow introduce these operators in the naive definition (13.1). This is the new element that must be added to obtain a growing entropy as announced in Section 9.

The most general and satisfactory solution is to define a projector

$$P: \quad \Phi_- \rightarrow \Phi_P \subset \mathcal{L} \tag{13.6}$$

and consider new projected density matrices $\tilde{\rho} = P\rho$ in such a way that a set of necessary properties should be fulfilled. These properties are:

1.

$$Tr\tilde{\rho}(t) = Tr\tilde{\rho}_* = 1 \Rightarrow Tr\tilde{\rho}_1(t) = 0 \tag{13.7}$$

in order for the new projected matrices $\tilde{\rho}$ to have the same normalization properties as the old ρ. In short: $Tr\rho = Tr\tilde{\rho}$. In this sense P is completely different than $\overline{M}$ or $\tilde{M}$ that transform a constant trace into a variable one.

2.

$$Tr\tilde{\rho}_1^2 \neq 0, \qquad \tilde{\rho}_1^n(t) \neq 0, \qquad m > 2 \tag{13.8}$$

and this must also be the case if some factors $\tilde{\rho}_*^{-1}$ are included in the product. From these properties we would obtain the growing of the entropy.

3.

$$\tilde{\rho}_* = P\rho_* = \rho_* \tag{13.9}$$

in such a way that in the limit $t \rightarrow \infty$ the projection P turns out to be irrelevant and we reobtain the usual Gibbs entropy for the equilibrium.

The most general P acting on Φ_- is

$$P = \sum_{ij} p_{ij} |\bar{i})(\tilde{j}| + \sum_{ij} q_{ij} |\tilde{i})(\tilde{j}| \tag{13.10}$$

where for the sake of simplicity we have once more trivialized the notation and we have written the bases $\{\beta(\overline{\cdot\cdot})\}$ and $\{\beta(\widetilde{\cdot\cdot})\}$ with just one index as $\{|\bar{i})\}$ and $\{|\tilde{i})\}$. the most general matrix of Φ_- reads

$$\rho = \sum_i \rho_i |\bar{i}) \tag{13.11}$$

namely Eq. (4.7) in the new notation. If $p_{ij}, q_{ij} \in \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}$, then the behavior at infinities is good enough and then $P\rho \in \Phi_P \subset \mathcal{L}$. Now if we symbolize the diagonal states by $|I)$ and the off-diagonal ghost by $|\overline{\mu})$, we have that

$$\rho = \sum_I \rho_I |I) + \sum_\mu \rho_\mu |\overline{\mu}) \tag{13.12}$$

and

$$P = \sum_{IJ} p_{IJ}|I)(J| + \sum_{I\mu} p_{I\mu}|I)(\tilde{\mu}| + \sum_{\mu I} p_{\mu I}|\overline{\mu})(I|$$
$$+ \sum_{\mu\nu} p_{\mu\nu}|\overline{\mu})(\tilde{\nu}| + \sum_{\mu I} q_{\mu I}|\tilde{\mu})(I| + \sum_{\mu\nu} q_{\mu\nu}|\tilde{\mu})(\tilde{\nu}| \tag{13.13}$$

But in order to satisfy property 3 one must have

$$P_{IJ} = \delta_{IJ}, \qquad p_{\mu I} = q_{\mu I} = 0 \tag{13.14}$$

Let us now compute

$$P\rho = \sum_{I} \rho_I|I) + \sum_{I\mu} p_{I\mu}\rho_\mu|I) + \sum_{\mu\nu} p_{\mu\nu}\rho_\nu|\overline{\mu}) + \sum_{\mu\nu} q_{\mu\nu}\rho_\nu|\tilde{\mu}) \tag{13.15}$$

and let us compute the traces of ρ and $P\rho$:

$$Tr\rho = \sum_{I} \rho_I, \qquad TrP\rho = \sum_{I} \rho_I + \sum_{I\mu} p_{I\mu}\rho_\mu \tag{13.16}$$

According to condition 1, these norms must be equal for every ρ_μ; thus $p_{I\mu} = 0$.

Also, as $P^2 = P$, one must have

$$\sum_{\lambda} p_{\mu\lambda}p_{\lambda\nu} = p_{\mu\nu} \sum_{\lambda} q_{\mu\lambda}p_{\lambda\nu} = q_{\mu\nu} \tag{13.17}$$

Then these are the conditions that the coefficients of the projector must satisfy. These equations have solutions, since the first one is satisfied for any projector in the "ghost" space and the second is satisfied if $p_{\mu\nu} = q_{\mu\nu}$. But, of course, more general solutions can be found. So the solution is not unique and we would have many possible projectors that will give rise to many possible entropies (out of equilibrium), as we will see. We also remark that the condition $P^2 = P$ is not strictly necessary, so we also can develop a theory where P is not a projector. Finally,

$$P\rho_1 = \sum_{\mu\nu} p_{\mu\nu}\rho_\nu|\overline{\mu}) + \sum_{\mu\nu} q_{\mu\nu}\rho_\nu|\tilde{\mu}) \tag{13.18}$$

Thus $Tr\tilde{\rho}_1^2 = Tr(P\rho_1)^2 \neq 0$, $\tilde{\rho}_1^n(t) \neq 0$, $n > 2$, and these results will be valid if we intercalate ρ_*^{-1} factors. In fact, the "q" term in P introduces $|\widetilde{\text{ghost}}\rangle$, which does not vanish when multiplied by the $|\text{ghost}\rangle$. Then condition 2 is also satisfied.

Now we can redefine our quantum conditional entropy as

$$S = -Tr[\tilde{\rho}\log(\tilde{\rho}\rho_*^{-1})] \tag{13.19}$$

Repeating the calculations, we obtain

$$S(t) = -\tfrac{1}{2}e^{-2\gamma_{\min}t}Tr[(\tilde{\rho}_1(t))^2\rho_*^{-1}(t)] + \cdots \tag{13.20}$$

where now the dots symbolize terms that vanish faster than the first one. But with this new definition $S(t) \neq 0$ and

$$\lim_{t\to\infty} S(t) = 0 \tag{13.21}$$

namely $S(t)$ goes to the equilibrium entropy when $t \to \infty$. So, from Eq. (13.20) we can say that near equilibrium our quantum conditional entropy is negative and it always grows, but of course, far from the equilibrium, the evolution may not have these properties.

So we realize that the ever-growing property of the entropy is not a quantum property, but just a classical one. Thus, for the sake of completeness we will sketch in the Appendix the relation between the classical case and the quantum case. Then, using our equations, we will see that the classical analogue of our quantum conditional entropy never decreases.

But for each P there is a different entropy, both in the quantum and the classical cases. This is not a problem since all these entropies coincide in the equilibrium limit, due to condition 3 (also near equilibrium all of them have the same dominant damping factor). So we have one and only one equilibrium thermodynamic entropy, the Gibbs one, and many nonequilibrium entropies, which depend on the choice of P, namely the choice of the apparatus that measures these entropies. So our position is that, even if the arrow of time is intrinsic and defined by space Φ_-, the definition of the out-of-equilibrium entropy is observer (or measurement apparatus)-dependent. But all these entropies grow in the same time direction and therefore share the same arrow of time.

14. THERMALIZATION

From Eq. (3.5) we know that our method allows us to split the Hamiltonian into two noninteracting parts, a discrete one related to the oscillating and damped modes of the eventual n oscillators of our model, and a continuous one, related to the field or the bath ω. In Eq. (3.5) there is no interaction between the discrete and continuous modes, so we may ask how a bath in thermal equilibrium can thermalize the oscillators in our model.

Obviously the thermalization can only be seen if we go to the basis where there is some interaction between the oscillators and the bath, namely the basis of the unperturbed Hamiltonian H_0, which we can diagonalize as

$$H_0 = \sum_n E_n^{(0)}\big|E_n^{(0)}\rangle\langle E_n^{(0)}\big| + \int_0^\infty E^{(0)}\big|E^{(0)}\rangle\langle E^{(0)}\big|\, dE^{(0)} \tag{14.1}$$

while the perturbed Hamiltonian reads

$$H = \sum_n z_n |\overline{f_n}\rangle\langle\tilde{f}_n| + \int_0^\infty \omega |\overline{f_\omega}\rangle\langle\tilde{f}_\omega| \, d\omega \tag{14.2}$$

But $H = H_0 + V$, so in the basis $\{|E_n^{(0)}, E^{(0)}\rangle\}$ we can see the interaction and the thermalization phenomena, but not in the basis of Eq. (14.2).

In fact, any initial ρ in a thermal bath can be written as

$$\rho = \sum_{nm} \rho_{nm} |E_n^{(0)}\rangle\langle E_m^{(0)}| + Z \int_0^\infty e^{-\beta E^{(0)}} |E^{(0)}\rangle\langle E^{(0)}| \, dE^{(0)} \tag{14.3}$$

(in this section we do not consider the off-diagonal components of the bath, since we consider that the bath is always in equilibrium, so these components take no part in the thermalization procedure). The unperturbed bath is a thermal state at temperature β^{-1} and Z is a normalization coefficient. Then, introducing

$$I = \sum_{n'} |\overline{f_{n'}}\rangle\langle\tilde{f}_{n'}| + \int_0^\infty |\overline{f_\omega}\rangle\langle\tilde{f}_\omega| \, d\omega \tag{14.4}$$

in each term and tracing away the bath, because we are only interested in the oscillator mode term ρ_O we have

$$\begin{aligned} \rho_O &= \sum_{nm} \rho_{nm} \sum_{n'm'} |\overline{f_{n'}}\rangle\langle\tilde{f}_{n'}|E_n^{(0)}\rangle\langle E_m^{(0)}|\overline{f_{m'}}\rangle\langle\tilde{f}_{m'}| \\ &\quad + Z \int_0^\infty e^{-\beta E^{(0)}} \sum_{n'm'} |\overline{f_{n'}}\rangle\langle\tilde{f}_{n'}|E^{(0)}\rangle\langle E^{(0)}|\overline{f_{m'}}\rangle\langle\tilde{f}_{m'}| \, dE^{(0)} \\ &= \sum_{n'm'} |\overline{f_{n'}}\rangle\langle\tilde{f}_{m'}| \Bigg[\sum_{nm} \rho_{nm} \langle\overline{f_{n'}}|E_n^{(0)}\rangle\langle E_m^{(0)}|\tilde{f}_{m'}\rangle \\ &\quad + Z \int_0^\infty e^{-\beta E^{(0)}} \langle\overline{f_{n'}}|E^{(0)}\rangle\langle E^{(0)}|\tilde{f}_{m'}\rangle \, dE^{(0)} \Bigg] \end{aligned} \tag{14.5}$$

To see what really is going on, let us go to the Friedrichs model of ref. 15 and consider the case where $V \to 0$ and $t \to \infty$, using equations (5.20) of that paper, which translated into our notation reads

$$\begin{aligned} &|\langle\overline{f_{n'}}|E_n^{(0)}\rangle|^2 \to 0, \qquad |\langle E_m^{(0)}|\tilde{f}_{m'}\rangle|^2 \to 0 \\ &|\langle\overline{f_{n'}}|E^{(0)}\rangle|^2 \to \sim\delta(E^{(0)} - E_{n'}^{(0)}), \qquad |\langle E^{(0)}|\tilde{f}_{m'}\rangle|^2 \to \sim\delta(E^{(0)} - E_{m'}^{(0)}) \end{aligned} \tag{14.6}$$

where we have neglected the terms like $|\langle \overline{f_{n'}}|E^{(0)}\rangle|^2$ since they vanish in the limit when $V \to 0$ because they relate the discrete eigenkets to the continuous ones. Using these equations, we obtain the result

$$\rho_0 \to \sim \sum_n |\overline{f_n}\rangle\langle \tilde{f}_n| e^{-\beta E_n^{(0)}} \sim \sum_n |E_n^{(0)}\rangle\langle E_n^{(0)}| e^{-\beta E_n^{(0)}} \tag{14.7}$$

so the oscillators will be thermalized when $t \to \infty$, for small interactions.

15. LOCAL AND GLOBAL TIME ASYMMETRY

Let us now make precise the meaning of two important words: *conventional* and *substantial*:

In mathematics we are used to working with identical objects, like points, the two directions of an axis, the two semicones of a light cone, the two time orientations of a time-oriented manifold, etc.

In physics there are also identical objects, like identical particles, the two spin directions, the two minima of a typical two-minimum potential, etc.

When [3, 32] we are forced to call *two identical objects by different names* we will say that we are establishing a *conventional difference*, e.g., when we call e_1 and e_2 two electrons, or "up" and "down" two spin directions, or "right" and "left" two minima of a symmetric potential curve; while if we call *two different objects by different names,* we will say that we are establishing a *substantial difference.*

The problem of time asymmetry is that, in all time-symmetric normal physical theories, usually the difference between past and future is just conventional. In fact, if we change the word "past" by the word "future" in these theories, the theory remains valid and nothing actually changes. But we have a clear psychological feeling that the past is substantially different from the future. Thus the problem of the arrow of time is to find theories where the past is substantially different from the future, such that the usual well-established physics remains valid. As we will see, our minimal irreversible quantum mechanics is one of these theories.

But up to now we have just postulated that the states $\rho \in \Phi_-$ and the observables $R \in \Phi_+$, namely we have established a local time asymmetry within a particular quantum system. To establish a global time asymmetry for the whole universe is quite a different task.

It is clear that if we content ourselves with local time asymmetry we face several problems; e.g., as Φ_- and Φ_+ are only conventionally different, there is no reason to justify that all the states would belong to a space Φ_- and all the operators to a space Φ_+ in all the systems of the universe. Most likely the states would belong to Φ_- in the 50% of the systems of the universe and to Φ_+ in the other 50%. For the operators we would have the inverse situation. Then what would happen when a system of the first class interacts with a system of the second class? Perhaps somehow would an average arrow of time be established?

To escape from this dilemma it is clear that we need a cosmological model that correlates all the local arrows of time, ad we know that current cosmological models are not completely satisfactory. Nevertheless, the most satisfactory, realistic, and simplest model to solve the problem is the Reichenbach global system, [33], which we will further explain.

16. REICHENBACH GLOBAL SYSTEM

The set of *irreversible* processes within the universe, each one beginning in an unstable nonequilibrium state, can be considered a *global system* [33, 34]. Namely, every one of these branching processes began in a nonequilibrium state such that this state was produced by a previous process of the set; e.g., the Gibbs ink drop (initial unstable state) spreading in a glass of water (irreversible process) may only exist if it was produced by an ink factory (since the probability to concentrate an ink drop by a fluctuation in a glass where the ink is mixed with the water, is extremely small). This factory extracted the necessary energy from an oven where coal (initial unstable state) was burning (branch irreversible process); in turn the coal was created with energy coming from the sun, where hydrogen (initial unstable state) was burning (branch irreversible process); finally, hydrogen was created using energy obtained from the unstable initial state of the universe (the absolute initial state of the global system). It is now clear that within the global system all the arrows of time point to the same direction, since all of them originated in the same global initial unstable state (why the initial state is unstable is explained in refs. 34 and 35 and it is not of concern in this paper).

Of course the Reichenbach global system was originally imagined as a model of the classical universe. But we can also imagine a quantum global system.

Let us draw the usual diagram of a scattering experiment (Fig. 2) to have a graphic idea of the nature of the unstable states. In the center of the diagram there is a black box that symbolizes any scattering process. A set of stable "in" states $a_1, a_2, \ldots$ is transformed by the scattering process into another set of stable "out" states $b_1, b_2, \ldots$. It is a reversible process because the evolution equations are time-reversible, so we can interchange the "in" and "out" states and all the results remain valid. In fact, Fig. 2 is essentially symmetric. The variation of the quantum entropy vanishes in this process.

Now, let us cut the black box into two parts by the dashed line draw at $t = 0$. Then, we can consider the right side of the figure, namely Fig. 3. This figure was introduced by Bohm [5], so we will call this kind of figure a

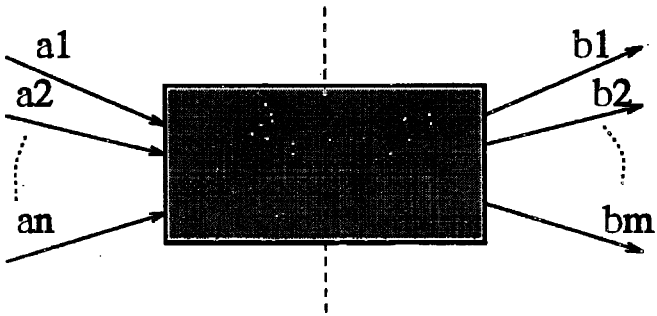

Fig. 2. Ordinary scattering diagram, with a black box.

Bohm diagram. In Fig. 3 the set of stable incoming states creates a set of unstable states, u_1, u_2, . . . , which are growing states and they belong to space ϕ_+ [7] (e.g., radiation exciting an electron of the ground state). As the states of ϕ_+ are linear combinations of the states of $\phi_-^\times$, in some sense they can also be considered as growing states and as such they can be symbolized as horizontal lines inside the half box. Figure 3 is asymmetric and it symbolizes an irreversible creation process. The evolution equation is still time-symmetric, but irreversibility is introduced by the growing nature of the states

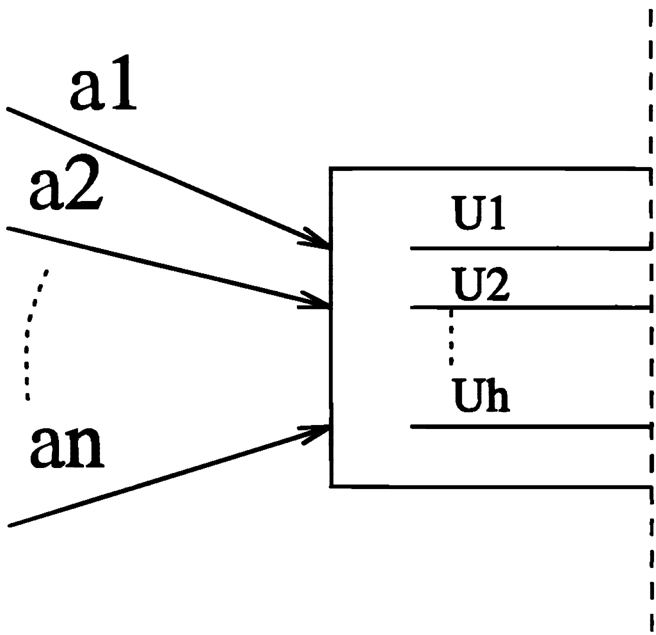

Fig. 3. Bohm diagram for growing states.

of space $\phi_-^\times$ or by the noninvertible time evolution operator acting on ϕ_- for $t < 0$. the variation of the entropy is negative in this process. This is not contradictory, since in every creation process there is an incoming energy and then we can consider the system as an open one.

We can also consider the second half of Fig. 1, namely Fig. 4. It is the Bohm diagram of a decaying process where a set of unstable decaying states u_1, u_2, . . . , which are linear combinations of the basis of space $\phi_+^\times$, is transformed into a set of stable outgoing stable states of ϕ_- (e.g., an excited electron decaying into the ground state and emitting radiation). Figure 4 is asymmetric and symbolizes a decaying irreversible process. Again, the evolution equations are still time-symmetric, but the decaying nature of the states of space $\phi_+^\times$ introduces irreversibility, etc. The variation of the entropy is positive in this process. These would be the diagrams corresponding to local processes.

But Bohm diagrams also allow us to see the quantum structure of a global system (Fig. 5). The universe is represented by a set of branching scattering processes with one initial unstable state symbolized by the cut box (at the Big-Bang time $t = 0$ at the far right. Each subsystem going from an unstable state to equilibrium (an ink drop spreading in water, a sugar lump dissolving in coffee, . . .) is symbolized by a decay process like the one of Fig. 4, namely the diagram in the upper shaded box of Fig. 5. The creation

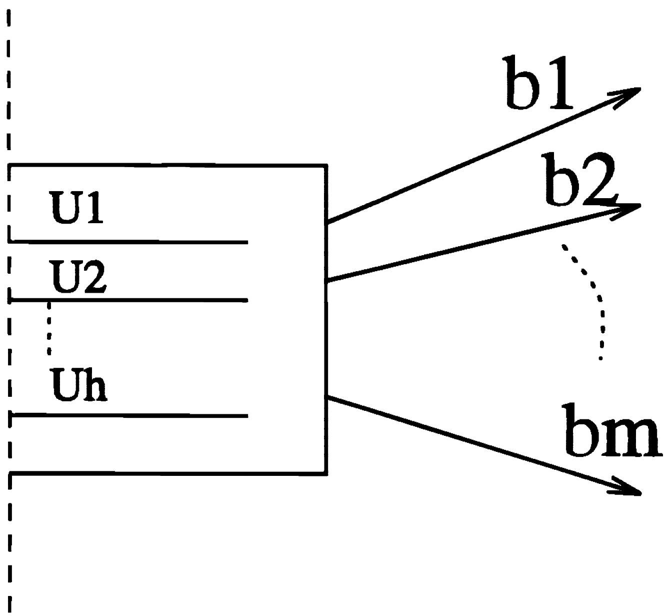

Fig. 4. Bohm diagram for decaying states.

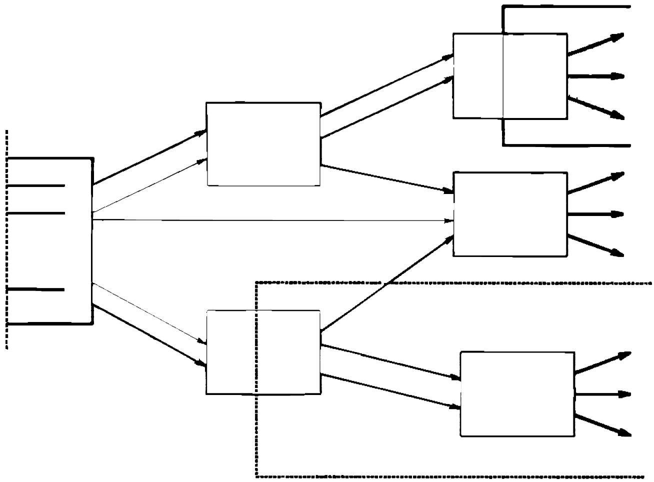

Fig. 5. Bohm diagram for the branch system.

of an unstable state is symbolized by a creation process (like the one of Fig. 3) where energy comes from a previous decay process (the ink factory with its oven). One of these larger subsystems with its source of energy is represented in the biggest lower dotted box in Fig. 5. The overall process is irreversible, because Fig. 5 is asymmetric, and if we have a model of this universe (see some simple models in refs. 25 and 36) the state of the universe must belong to some global space ϕ_-^G or Φ_-^G. Therefore in this diagram there is a clear arrow of time. but in the previous diagrams (Fig. 3 or Fig. 4) the arrow of time was a "local" one, while this diagram has one of the most important characteristics of the observed time asymmetry: it is global. This is the way to introduce the arrow of time in a time-symmetric dynamical formalism: by a global and generalized symmetry-breaking process.

But we must remember that the difference between the global Φ_-^G and the global Φ_+^G of the whole universe global system is just conventional since these two spaces are identical. Thus physics is the same in Φ_-^G as in Φ_+^G. Think of a cosmological model (Fig. 5), life will be the same in this universe (with a quantum state in space Φ_-^G) as in the universe of Fig. 6, the time-inverted image of Fig. 5 (with a t-inverted quantum state in space Φ_+^G). In fact, since in both models of the universe (if completely computed) all the arrows of time must point in the same direction, there is no physical way to decide if we are in one model or the other. So both models are identical. Thus the choice between Φ_-^G and Φ_+^G is just conventional and physically irrelevant (as in choosing one of the two minima of the potential in the spontaneous symmetry-breaking problem).

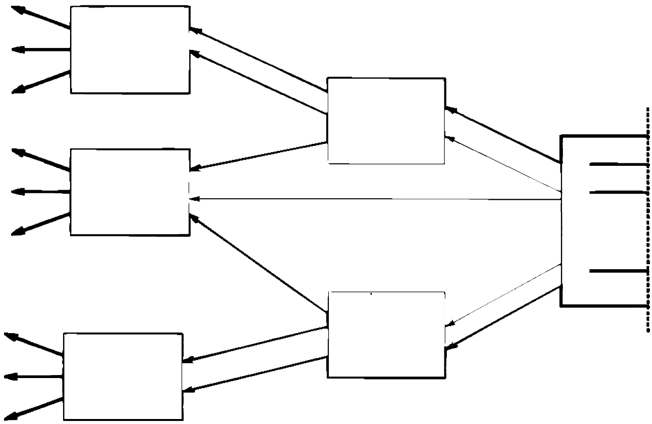

Fig. 6. Specular image of Fig. 5.

But once this choice is made, a substantial difference is established in the model e.g., the only time evolution operator is $U_{-}(t) = e^{-iHt}$, $t > 0$, and it cannot be inverted; we only have equilibrium toward the future, entropy growing, etc.

Once the Φ_{-}^{G} or the Φ_{+}^{G} is chosen in the global system, we are forced to choose the corresponding spaces in the local subsystems, if we want to study these subsystems as isolated systems, and a global arrow of time is established. This fact solves the problem stated in the preceding section.

Thus the choice between Φ_{-}^{G} and Φ_{+}^{G} is trivial and unimportant (but this choice must be a global one), which is why the arrow of time is not introduced by hand in our theory. The important choice is between $\mathcal{L}^{G}$ and Φ_{-}^{G} (or Φ_{+}^{G}) as the space of our physically admissible states. And we are free to make this choice, since a good physical theory begins with the choice of the best mathematical structure to mimic nature. *Thus, our thesis is essentially that time-asymmetric mathematical structures like ours mimic in a more economical way the time asymmetry of the nature in which we live than do time-symmetric mathematical structures.*

17. OTHER RESULTS

The main results related to quantum mechanics are stated in the above sections. But we must comment that using the present formalism, all the relevant results of irreversible statistical mechanics can also be obtained, e.g., all the results of ref. 37, as is proved in ref. 2, because the main Π projector of ref. 37 can be defined using Gel'fand triplets. Also, in some simple cases, we can go from quantum models to classical ones [11], where we find the same philosophy in the classical cases. Chaotic models like the Baker's transformation and Renyi maps are also treated with the same method, with good results [31, 38]. Other interesting results are contained in refs. 5, 16, 21, 39, and 40. Thus, the given quantum axiomatic formalism is a general method for dealing with irreversible processes.

18. CONCLUSION

Our axiomatic formalism condenses the main ideas of how to solve the problem of time asymmetry pioneered by various authors as cited in the references. This method, based on the reasoning given in the introduction, yields correct physical results that coincide with those obtained by other methods (coarse-graining, traces, stochastic noise, loss of information, etc.).

We do not foresee any crucial experiment to settle which of the quoted methods (coarse-graining, traces, stochastic noise, loss of information, etc.) or ours is the correct one, because we think that they are somehow complementary and that they only explain the real physical world in different ways. Therefore we believe that we must solve at least three problems in order to complete our theory.:

(a) The methods of coarse-graining, traces, projections, stochastic noise, loss of information, etc., have clear physical motivations. On the contrary, our method is just based on the search for the minimal mathematical structure to explain time irreversibility. Even if this reason were sufficient for mathematically minded readers, it turns out to be not so eloquent for physically minded ones. So we must find a relation among all the methods because most probably they are all based on the same or very similar (eloquent) physical grounds. This work was already begun in ref. 10, where it is shown that causality is the real reason for the choice of the proper space of physical states.

(b) We have admitted, under Eq. (2.9), that there is not a unique way to define the space ϕ_-. We must find the necessary and sufficient condition to define a unique space ϕ_- or at least a class of spaces all of them endowed with enough properties to explain all phenomena related with time asymmetry. Lax–Phillips scattering theory, recently redeveloped by Pavlov [42], and ref. 10 seem the more promosing ways to solve the problem.

(c) Essentially space Φ_- is the one where all out-of-equilibrium entropies turn out to be growing. But as we said at the beginning of Section 13, a complete study that singles out in an indisputable way an entropy that generalizes the phenomenological thermodynamic entropy at equilibrium is missing. Most likely only when this task will be accomplished will we be able to motivate the choice of the space Φ_- convincingly.

Meanwhile we just claim that our formalism is a contribution to the understanding of time asymmetry.

APPENDIX. WIGNER FUNCTION INTEGRAL AND CLASSICAL ENTROPY

Let ρ be a density matrix of Liouville space $\mathcal{L} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ and let $\{|q\rangle\}$ be the configuration or position basis of the Hilbert space $\mathcal{H}$. The Wigner function corresponding to state ρ is a real, but not positive-definite quantity that reads [ref. 41, Eq. (2.1)]

$$\rho_W(q, p) = \pi^{-1} \int \langle q - \lambda|\rho|q + \lambda\rangle e^{2i\lambda p}\, d\lambda \tag{A.1}$$

This equation is valid if the wave function has only one variable. If it has n variables the π^{-1} must be changed by π^{-n}. It can be proved that

$$L\rho_{\mathrm{W}}(q, p) = \pi^{-1} \int \langle q - \lambda|L\rho|q + \lambda\rangle e^{2i\lambda p}\, d\lambda + O(\hbar) \tag{A.2}$$

where L is respectively the classical and quantum Liouville operator. In the classical limit $\hbar \to 0$, therefore, ρ_{W} can be considered as the classical distribution function corresponding to ρ. As in the classical regime we practically work in this limit, we will consider that Eq. (A.1) is the relation between the quantum density matrix and the classical distribution function. In fact, even if ρ_{W} is not generally positive definite, using the Wigner integral, we can obtain quantum equations from classical equations and vice versa, as we will see in a few examples. For example, let us observe that [ref. 41, Eq. (2.6)]

$$\begin{aligned} \|\rho_{\mathrm{W}}\| &= \iint \rho_{\mathrm{W}}(q, p)\, dq\, dp \\ &= \int dq \int \langle q - \lambda|\rho|q + \lambda\rangle\, \delta(\lambda)\, d\lambda = Tr\rho \end{aligned} \tag{A.3}$$

so to the classical norm corresponds the quantum trace. Also, if we define the classical analogue of operator O as [ref. 41, Eq. (2.12)]

$$O_{\mathrm{W}}(q, p) = \int dz\, e^{ipz} \langle q - \tfrac{1}{2}z|O|q + \tfrac{1}{2}z\rangle \tag{A.4}$$

we obtain that

$$(\rho_{\mathrm{W}}|O_{\mathrm{W}}) = \iint \rho_{\mathrm{W}}(q, p) O_{\mathrm{W}}(q, p)\, dp\, dq = Tr(\rho O) \tag{A.5}$$

Therefore to the inner product in classical Liouville space corresponds the inner product in the quantum Liouville space. Finally, if O_1 and O_2 are two operators and $O_P = O_1O_2$, we have [ref. 41, Eq. (2.59)]

$$O_{P\mathrm{W}}(q, p) = Q_{1\mathrm{W}}(q, p)e^{\Lambda/2i}O_{2\mathrm{W}}(q, p) \tag{A.6}$$

where

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} \tag{A.7}$$

Therefore if $O_1 = O_2$, we have that

$$O_{P\mathrm{W}}(q, p) = [O_{1\mathrm{W}}(q, p)]^2 \tag{A.8}$$

This fact completes the analogy between classical and quantum spaces implemented by the Wigner integral.

As the Wigner integral (A.1) is a linear mapping, the quantum evolution equation (13.2) can be reproduced in the "classical case" as

$$\rho_{\mathrm{W}}(t) = \rho_{*\mathrm{W}} + e^{-\gamma_{\min}t}\rho_{1\mathrm{W}}(t) \tag{A.9}$$

where, somehow, ρ_{W} could be considered as a classical distribution function. This is not so because it is not positive definite, but we will see that ρ_{W} has this property near equilibrium, where we know that, due to the appearance of decoherence and correlations, the state is classical.

In fact, the classical states are defined as those having decoherence and correlations (this fact can be seen in refs. 30 and 23 using the coarse-graining method or using our method, in Section 11 of this paper for decoherence and in ref. 25 for correlations), namely those that can be expanded as

$$\rho = \sum_I \rho_I \big|I\rangle_{\mathrm{cor}}\langle I\big|_{\mathrm{cor}}, \qquad \rho_I \geq 0 \tag{A.10}$$

where $\big|I\rangle_{\mathrm{cor}}$ is a no-ghost state such that its position $\langle q\big|I\rangle_{\mathrm{cor}}$ and its momentum $\langle p\big|I\rangle_{\mathrm{cor}}$ are correlated, i.e., they are as defined as possible (e.g., as the ground state of a H atom). Precisely, they are as defined as is allowed by the uncertainty principle around a point (q_I, p_I). The quantum equilibrium state ρ_* is one of these states. Let $\rho_{*\mathrm{W}}$ be the classical equilibrium state. The Wigner integral of this state gives

$$\begin{aligned}\rho_{*\mathrm{W}}(q, p) &= \pi^{-1}\int_{-\infty}^{+\infty}\langle q - \lambda\big|\rho_*\big|q + \lambda\rangle e^{-2i\lambda p}\, d\lambda \\ &= \pi^{-1}\sum_I \rho_{I*}\int_{-\infty}^{+\infty}\langle q - \lambda\big|I\rangle_{\mathrm{cor}}\langle I\big|_{\mathrm{cor}}\big|q + \lambda\rangle e^{-2i\lambda p}\, d\lambda\end{aligned}$$

$$\cong \pi^{-1} \sum_I \rho_{I*} \int_{-\infty}^{+\infty} \langle q_I - \lambda | I \rangle_{\text{cor}} \langle I |_{\text{cor}} | q_I + \lambda \rangle \delta(\lambda) e^{-2i\lambda p} \, d\lambda$$

$$= \pi^{-1} \sum_I \rho_{I*} \int_{-\infty}^{+\infty} \left| \langle q_I | I \rangle_{\text{cor}} \right|^2 e^{-2i\lambda p} \delta(\lambda) \, d\lambda$$

$$= \pi^{-1} \sum_I \rho_{I*} \left| \langle q_I | I \rangle_{\text{cor}} \right|^2 \geq 0 \tag{A.11}$$

since the functions $\langle q_I | I \rangle_{\text{cor}}$ and $\langle I |_{\text{cor}} | q_I \rangle$ vanish except around a definite value of q_I, so the states $\langle q \pm \lambda |$ are only different from zero when $q \cong q_I$ and $\lambda \simeq 0$ [this is the reason for the factor $\delta(\lambda)$] and because of Eq. (7.4), which is valid for ρ_*. Then we conclude that the equilibrium states and the quantum states near (q_I, p_I) have the property $\rho_W(q, p) \geq 0$ and therefore these Wigner functions can be considered classical states. These states disappear if we wait long enough due to the damping term in Eq. (A.9).

So, in this classical regime, a usual conditional entropy can be defined, and any quantum formula can be reobtained in the classical case. Of course we can directly work in this regime if we consider the classical analogue of our formalism and the poles of the classical Liouville operator [31]. But now we know [20, 43] that this classical conditional entropy never decreases. So we can consider ρ_W and make a projection obtaining $\tilde{\rho}_W$ and define

$$S(\tilde{\rho}_W) = \int_\Gamma \tilde{\rho}_W \log \frac{\tilde{\rho}_W}{\rho_{*W}} \, dq \, dp \tag{A.12}$$

As this entropy is never decreasing, we know that $S(\dot{\tilde{\rho}}_W) \geq 0$. Also, as ρ_W is given by the sum of local terms (A.10), this definition contains the notion of local equilibrium entropy. Now we can repeat everything we did in the quantum case, since conditions 1–3 can be translated to the quantum case, condition 1 using Eq. (A.3), condition 2 using Eq. (A.6), and condition 3 using Eq. (A.1). So we obtain

$$S(\tilde{\rho}_W) = -\frac{1}{2} e^{-2\gamma_{\min} t} \int_\Gamma \frac{\tilde{\rho}_{1W}^2}{\rho_{*W}} \, dq \, dp + \cdots \tag{A.13}$$

where the dots symbolize higher order terms. Thus

$$\dot{S}(\tilde{\rho}_W) = \gamma_{\min} e^{-2\gamma_{\min} t} \int_\Gamma \frac{\tilde{\rho}_{1W}^2}{\rho_{*W}} \, dq \, dp + \cdots \tag{A.14}$$

So now it is clear that $\dot{S}(\tilde{\rho}_W) > 0$ and that we have obtained the second law of thermodynamics. This is not at all surprising since it is well known that, by a projection, the constant $S(\rho_W)$ can be transformed into the variable $S(\tilde{\rho}_W)$. We have just shown how our method works and also how we can obtain the classical results (A.9) and (A.13).

So our $S(\tilde{\rho}_W)$ has all the properties that we announced and we can claim that it is a good candidate to play the role of internal entropy.

ACKNOWLEDGMENTS

This work was partially supported by grants CI1*-CT94-0004 and PSS*-0992 of the European Community, PID-0150 of CONICET (National Research Council of Argentina), EX-198 of the Buenos Aires University, and 12217/1 of Fundación Antorchas, as well as grants from the Fondation pour la Receherche Fondamentale OLAM and the British Council.

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